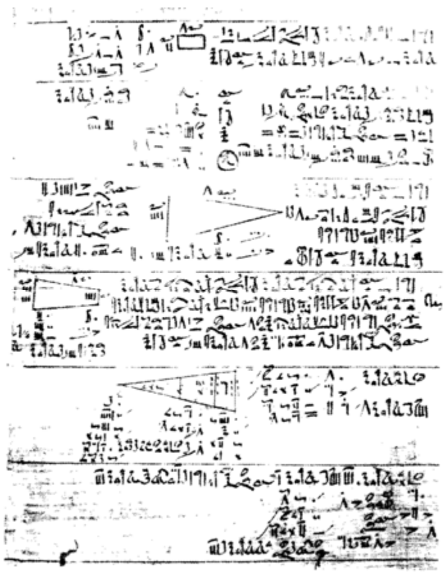
The Proto-Trigonometry of the Pyramids

By James D. Nickel



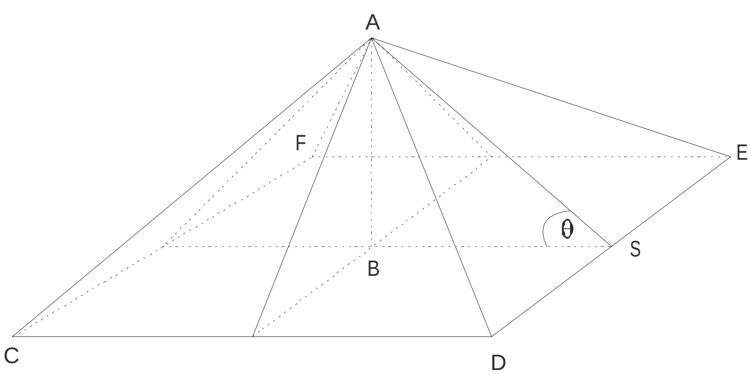
A portion of the Rhind Papyrus (Public Domain)

The Rhind Papyrus, a collection of 84 mathematical problems, was purchased in 1858 by the Scottish lawyer and antiquarian Alexander Henry Rhind (1833-1863).[1] It dates to around 1650 BC.

The Great Pyramid, Source: iStockPhoto

A section of these problems are pyramid-oriented and use the word *seked*. Egyptian pyramids had a square base and the copyist, self-identified as the scribe Ahmes, posed the following problem: The height of a pyramid is 250 cubits and its base is 360 cubits, what is its *seked*?

Ahmes then gives the solution and by it, we discover that the *seked* thus calculated is the "measure" of an angle, θ in the diagram below. Measure is in quotes and we shall see why in a moment. Technically, *seked* (or θ) is, using modern trigonometry, the *cotangent* of the angle between the base of the pyramid and one of its faces.



Seked

In the figure at right, we note the right triangle ABS where AB is the height and BS is half the length of the base. Ahmes is obviously using this right triangle because the first thing he does in his solution is to calculate half of the base; i.e., $\frac{1}{2}$ of 360 = 180.

Since the tangent ratio in this triangle is $\frac{AB}{BS}$, the *co*mplementary ratio, or *co*tangent, is $\frac{BS}{AB}$. Two angles are *complementary* if they sum to 90°. In a right triangle, its two acute triangles are always complementary. This means that $\tan\theta = \cot(90° - \theta)$, where $90° - \theta = \angle BAS$, and $\cot\theta = \tan(90° - \theta)$. In the right triangle, $\tan\theta = \frac{AB}{BS}$ and $\cot\theta = \frac{BS}{AB}$. Hence, there is a reciprocity relationship between the tangent and cotangent ratios.

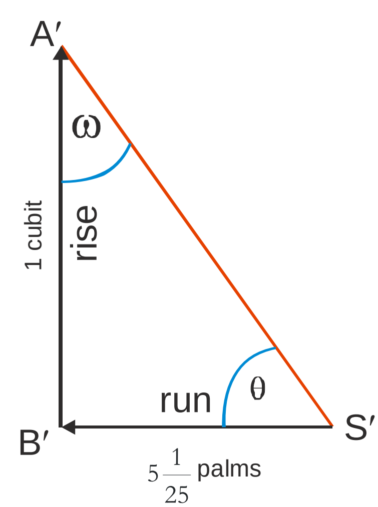
[1] The source for much of the information in this essay is the excellent book by Eli Maor, *Trigonometric Delights* (Princeton: Princeton University Press, 1998).

In a round about way, Ahmes calculates the cotangent ratio of BS to AB or $\frac{BS}{AB} = \frac{180}{250} = \frac{18}{25}$. Since Egyptians only used unit fractions (fractions whose numerator is 1), Ahmes wrote $\frac{18}{25}$ as $\frac{1}{2} + \frac{1}{5} + \frac{1}{50}$. Next, Ahmes multiplied his answer by 7. Why? Ancient Egyptians builders measured horizontal distances in *palms* (think of the "palm" of the hand and note that the "hand" was also a unit of Egyptian measure) and vertical distances in *cubits*.[2] For these ancient architects, 1 cubit = 7 palms. We note that $\frac{18}{25}\cancel{\text{cubits}} \times \frac{7\text{ palms}}{\cancel{\text{cubit}}} = 5\frac{1}{25}\text{ palms}$. To these builders $5\frac{1}{25}$ represented a ratio, not a number. In terms of the right triangle, it represented what modern builders call a *batter*[3], a run-to-rise (horizontal-to-vertical) *slope*.[4]

The builders of the Egyptian pyramids used the run-to-rise ratio to ensure a constant slope for each face of the pyramid. In Ahmes' example, the builders would have used the $\frac{5\frac{1}{25}\text{ palms}}{1\text{ cubit}}$ ratio. This means that for every $5\frac{1}{25}$ palms of horizontal distance, they made sure that the equivalent vertical distance was *1 cubit* (the figure below illustrates this procedure). By this method, these builders guaranteed that the right triangle A′B′S′ would be *similar* to the right triangle ABS (or, in symbols, $\Delta A'B'S' \sim \Delta ABS$) and thereby each face of the pyramid would maintain the same *seked*.

In summary, the concept of angle measurement, as we know it today, does not appear in Egyptian writings. They were *not* dealing with quantitative trigonometry; *they were only dealing with run-to-rise ratios.* In this sense, though, their work foreshadowed trigonometry; it was a proto-trigonometry.

What is θ? Since $\cot\theta = \frac{18}{25} = 0.72$, then $\theta = \cot^{-1}(0.72)$. Most modern calculators do not have a function to find the inverse cotangent so we will use the complement of the cotangent, the tangent. We set $\omega = 90° - \theta$. Hence, $\omega = \tan^{-1}(0.72) \Rightarrow \omega = 35.75°$ (the complement of the angle we are after). Since $\omega = 90° - \theta$, then $\theta = 90° - \omega$. Hence, $\theta = 90° - 35.75° = 54.25°$. This measurement corresponds closely to the actual "face to base" angles of the pyramids of Egypt. So, Ahmes the scribe knew of what he wrote.



Run-to-rise

[2] Ahmes disregarded this distinction when, in his problem, he gave the measurement of the base of the pyramid, a *horizontal* distance, in cubits.
[3] Technically, a *batter* is the backward and upward slope of the face of a wall.
[4] The cotangent ratio is a run-to-rise slope while the tangent ratio is a rise-to-run slope.