PROVING THE ASSOCIATIVE LAW OF ADDITION

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The Associative Law of Addition states that for any real number a, b, c, grouping binary operations does not matter, or $a + (b + c) = (a + b) + c$

To prove this law, we invoke the dance of Mathematical Induction.

Step 1. We first must prove this case when $c = 1$, i.e., $a + (b + 1) = (a + b) + 1$

Consider the operation $a + b$. We can understand it this way: Given a, we add 1 to it, then add another 1 to it until we add 1 to it b times. We write:

$$a + b = a + \underbrace{1 + 1 + \ldots + 1}_{b \text{ times}}$$

For example, if $b = 6$, we write:

$$a + b = a + \underbrace{1 + 1 + 1 + 1 + 1 + 1}_{6 \text{ times}}$$

What does $a + (b + 1)$ mean? To a, we add $b + 1$ successive additions of 1 to a. We write:

$$a + (b + 1) = a + \underbrace{1 + 1 + \ldots + 1}_{b \text{ times}} + 1$$

Note: $a + \underbrace{1 + 1 + \ldots + 1}_{b \text{ times}} + 1 = (a + b) + 1$

We conclude: $a + (b + 1) = (a + b) + 1$

We have, therefore, proved this case when $c = 1$.

Step 2. Next, we assume the law is true for any value of c, i.e., n, or

$a + (b + n) = (a + b) + n$

We add 1 to both sides, grouping $a + (b + n)$ and $(a + b) + n$ in brackets:

$a + (b + n) = (a + b) + n \Leftrightarrow [a + (b + n)] + 1 = [(a + b) + n] + 1$

We know: $a + (b + 1) = (a + b) + 1$

To avoid confusion in our subsequent analysis, let's rewrite what we know changing a and b to x and y respectively. We get:

$x + (y + 1) = (x + y) + 1$

We first consider the expression $[(a + b) + n] + 1$

If we replace $a + b$ with x and n with y, then $[(a + b) + n] + 1 = (x + y) + 1$.

But we know that $(x + y) + 1 = x + (y + 1)$. Therefore, returning x to $a + b$ and n to y, we get:

$[(a + b) + n] + 1 = (a + b) + (n + 1)$

We now consider the expression $[a + (b + n)] + 1$

If we replace a with x and $b + n$ with y, then $[a + (b + n)] + 1 = (x + y) + 1$.

But we know that $(x + y) + 1 = x + (y + 1)$. Therefore, returning x to a and $b + n$ to y, we get:

$[a + (b + n)] + 1 = a + [(b + n) + 1]$

We can now rewrite $[a + (b + n)] + 1 = [(a + b) + n] + 1$ this way:

$[a + (b + n)] + 1 = [(a + b) + n] + 1 \Leftrightarrow a + [(b + n) + 1] = (a + b) + (n + 1)$

Investigating the expression $(b + n) + 1$, we can rewrite it as follows:

$(b + n) + 1 = b + (n + 1)$

Therefore, we conclude:

$a + [(b + n) + 1] = (a + b) + (n + 1) \Leftrightarrow a + [(b + (n + 1))] = (a + b) + (n + 1)$

We note that the equation $a + [(b + (n + 1))] = (a + b) + (n + 1)$ is $a + (b + c) = (a + b) + c$ for $c = n + 1$

Therefore, given that the proposition is true for some number n, we have reasoned that it is necessarily true for the successor of n, i.e., $n + 1$. By the dance of Mathematical Induction, we have proved the Associative Law of Addition.

QED