THEOREM: For a positive integer N that is not a perfect square, $\sqrt{N}$ is irrational.

Source: *American Mathematical Monthly*, 115:6 (June-July 2008), p. 524.

The following analysis is an elegant, and therefore *beautiful* proof of this theorem. It was submitted for publication by Geoffrey C. Berresford, Department of Mathematics, Long Island University, Brookville, NY 11548.

I have taken the liberty to amplify some parts of this proof for purposes of perspicuity.

Method of proof: *reductio ad absurdum*

Given: N is *not* a perfect square.
Prove: $\sqrt{N}$ is irrational.

We assume $\sqrt{N}$ is rational.

Hence, $\sqrt{N} = \frac{a}{b}$ where a, b $\in +\mathbb{Z}$ and a, b are relatively prime (GCF, or Greatest Common Factor = 1).

$\sqrt{N} = \frac{a}{b} \Rightarrow N = \frac{a^2}{b^2}$.

Note that $\dfrac{\frac{a^2}{b^2} \cdot b}{a} = \dfrac{\frac{a^2}{b}}{a} = \frac{a^2}{b} \cdot \frac{1}{a} = \frac{a}{b}$.

Hence, $\sqrt{N} = \frac{a}{b} = \frac{Nb}{a}$.

Logical analysis:

If $\frac{a}{b} = \frac{Nb}{a}$ where $\frac{a}{b}$ is in lowest terms (GCF = 1), then the numerator and denominator of $\frac{Nb}{a}$ must be a common *integer* multiple, let's say *c*, of the numerator and denominator of $\frac{a}{b}$. For example, $\frac{3}{5} = \frac{6}{10} \Rightarrow \left(\frac{3}{5}\right)\left(\frac{2}{2}\right) = \frac{6}{10}$. Hence, $\frac{Nb}{a} = \left(\frac{a}{b}\right)\frac{c}{c}$. Since $\frac{Nb}{a} = \left(\frac{a}{b}\right)\frac{c}{c}$, then a = bc. Since a = bc, then, by substitution, $\frac{N\cancel{b}}{\cancel{b}c} = \left(\frac{\cancel{b}\cancel{c}}{\cancel{b}}\right)\frac{c}{\cancel{c}} \Leftrightarrow \frac{N}{c} = c \Leftrightarrow N = c^2$. Hence, N is a perfect square. But, this conclusion contradicts the given (N is *not* a perfect square). Therefore, if N is a positive integer that is not a perfect square, then $\sqrt{N}$ is irrational.
QED.