# POWER SEQUENCES

BY JAMES D. NICKEL

In an *arithmetic sequence*, a common "difference" separates each term in the sequence. Using function notation, every arithmetic sequence is a linear function of the form y = f(x) = ax + b (where the domain is the positive integers and *a* is the common difference). In a *geometric sequence*, you calculate each successive term by *multiplying by the same number*. Using function notation, every geometric sequence is an exponential function of the form $y = f(x) = ar^{x-1}$ (where the domain again is the positive integers and *r* is the common ratio or multiplier). In a *power sequence*, we calculate each successive term by raising consecutive positive integers to the same power.

Two common power sequences are the sequence of squares and the sequence of cubes (domain: positive integers). As functions, the sequence of squares is $y = f(x) = x^2$ and the sequence of cubes is $y = f(x) = x^3$. Both the sequence of squares and the sequence of cubes relate to dimensional analysis, two- and three-dimensions respectively. Hence, you can picture the sequence of squares as *area* and the sequence of cubes as *volume*.

## SEQUENCE OF SQUARES

The sequence of squares gives you progressively larger areas of a square as the length of one of its sides increases according to the sequence of positive integers; i.e., 1, 2, 3, 4, 5, …

| | | | | | | |
|---|---|---|---|---|---|---|
| Positive integers squared | $1^2$ | $2^2$ | $3^2$ | $4^2$ | $5^2$ | … |
| Sequence of Squares | 1 | 4 | 9 | 16 | 25 | … |

Remember, *squaring a number* is multiplying the number by itself; e.g., $2^2$ (read "2 squared") = 2 × 2 = 4. The superscript in $4^2$ is called the exponent**.** The exponent of $4^2$ is 2. The exponent of $10^3$ is 3. The value of $4^2 = 16$ is called the "second power of 4." *Exponentiation* is the process of raising a number to a power.

Every arithmetic operation has its inverse (e.g., the inverse of addition is subtraction and the inverse of multiplication is division); one "undoes" the other. The inverse of exponentiation is called extracting roots. For the sequence of squares, the inverse is called "extracting the square root" or "taking the square root" or "finding the second root of a number." The symbol for this operation is the radical sign, $\sqrt{\ }$, from the Latin *radix* meaning "root" or "radish."

In general, $\sqrt{a} = \pm n$. The ± ("plus or minus") symbol means there are two possible answers. For example, $\sqrt{4} = +2, -2$ because (+2) × ( +2) = 4 and (-2) × ( -2) = 4. The two solutions are written as follows: $\sqrt{4} = \pm 2$. The radical sign $\sqrt{\ }$ without a leading sign means we are asking *only for the positive root*. Think for a moment: Can you take the square root of a negative number; i.e., $\sqrt{-4}$ ? This will lead you into the mathematical realm of the "imaginary."

Here are some examples of taking the square root (positive root) of a few numbers.

- $\sqrt{1} = 1$
- $\sqrt{4} = 2$
- $\sqrt{9} = 3$
- $\sqrt{16} = 4$
- $\sqrt{25} = 5$

We can generate a handy "Table of Squares" for the first sixty positive integers.

| *Number* | *Square* | *Number* | *Square* | *Number* | *Square* |
|---|---|---|---|---|---|
| 1 | 1 | 21 | 441 | 41 | 1681 |
| 2 | 4 | 22 | 484 | 42 | 1764 |
| 3 | 9 | 23 | 529 | 43 | 1849 |
| 4 | 16 | 24 | 576 | 44 | 1936 |
| 5 | 25 | 25 | 625 | 45 | 2025 |
| 6 | 36 | 26 | 676 | 46 | 2116 |
| 7 | 49 | 27 | 729 | 47 | 2209 |
| 8 | 64 | 28 | 784 | 48 | 2304 |
| 9 | 81 | 29 | 841 | 49 | 2401 |
| 10 | 100 | 30 | 900 | 50 | 2500 |
| 11 | 121 | 31 | 961 | 51 | 2601 |
| 12 | 144 | 32 | 1024 | 52 | 2704 |
| 13 | 169 | 33 | 1089 | 53 | 2809 |
| 14 | 196 | 34 | 1156 | 54 | 2916 |
| 15 | 225 | 35 | 1225 | 55 | 3025 |
| 16 | 256 | 36 | 1296 | 56 | 3136 |
| 17 | 289 | 37 | 1369 | 57 | 3249 |
| 18 | 324 | 38 | 1444 | 58 | 3364 |
| 19 | 361 | 39 | 1521 | 59 | 3481 |
| 20 | 400 | 40 | 1600 | 60 | 3600 |

As we go from "number to square," we are squaring. We call these numbers *perfect squares.* The reverse, from "square to number," is taking the square root. Since mathematics is all about finding patterns, we can find several patterns in this table:

| *Last digit of number* | *Last digit of square* |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |
| 4 | 6 |
| 5 | 5 |
| 6 | 6 |
| 7 | 9 |
| 8 | 4 |
| 9 | 1 |

| *Last digit of square* | *Last digit of square root of the square* |
|---|---|
| 0 | 0 |
| 1 | 1, 9 |
| 4 | 2, 8 |
| 5 | 5 |
| 6 | 4, 6 |
| 9 | 3, 7 |

Notice the missing digits: 2, 3, 7, 8. This means that any numbers that end in these digits are *not* perfect squares (technically, these numbers are called *irrational numbers* and there are quite a few). $\sqrt{3}$, $\sqrt{13}$, $\sqrt{57}$, $\sqrt{112}$ and $\sqrt{88}$ are all examples of irrational numbers.

You can use the patterns revealed in this table to determine the *tens-place digit* for the square root of any number between 0 and 10,000. First, we note:

| | |
|---|---|
| $\sqrt{100}$ | 10 |
| $\sqrt{400}$ | 20 |
| $\sqrt{900}$ | 30 |
| $\sqrt{1600}$ | 40 |
| $\sqrt{2500}$ | 50 |
| $\sqrt{3600}$ | 60 |
| $\sqrt{4900}$ | 70 |
| $\sqrt{6400}$ | 80 |
| $\sqrt{8100}$ | 90 |
| $\sqrt{10000}$ | 100 |

Consider $\sqrt{2025}$. You know that the square root is between 40 and 50. Looking at the last digit tells you that if 2025 is a perfect square, its square root will end in 5. Hence, $\sqrt{2025} = 45$.

Consider $\sqrt{1024}$. You know that the square root is between 30 and 40. Looking at the last digit tells you that if 1024 is a perfect square, its square root will end in 2 or 8. Your answer is either 32 or 38. You know that $40^2 = 1600$ so the answer should be 32. Hence, $\sqrt{1024} = 32$

Consider $\sqrt{2187}$. You know that the square root is between 40 and 50. Since 2187 ends in 7, you know for sure that the square root is irrational.

Consider $\sqrt{4096}$. You know that the square root is between 60 and 70. Looking at the last digit tells you that if 4096 is a perfect square, its square root will end in either 4 or 6. Answer: 64 or 66. 4096 is closer to 3600 than to 4900 so the answer should be 64. Hence, $\sqrt{4096} = 64$

Consider $\sqrt{7529}$. You know that the square root is between 80 and 90. Since 7529 ends in 9, you know that if 7529 is a perfect square, its square root will end in 3 or 7. Answer: 83 or 87. 7529 is closer to 8100 than to 6400. So, the answer is 87. But, $87^2 = 7569$! 7529 is *not* a perfect square, and its square root is irrational. The fact that a number ends in a digit *other than* 2, 3, 7, 8 does not guarantee that the number is a perfect square.

Here is a wonderful number pattern involving the sequence of squares. First, note the connection between square numbers and the sum of odd numbers:

- The sum of the *first* odd number: $1 = 1 = 1^2$
- The sum of the first *two* consecutive odd numbers: $1 + 3 = 4 = 2^2$
- The sum of the first *three* consecutive odd numbers: $1 + 3 + 5 = 9 = 3^2$
- The sum of the first *four* consecutive odd numbers: $1 + 3 + 5 + 7 = 16 = 4^2$
- The sum of the first *five* consecutive odd numbers: $1 + 3 + 5 + 7 + 9 = 25 = 5^2$

In general, the sum of the first *n* consecutive odd numbers is $n^2$. Ancient Greek mathematicians first noted this pattern (they visualized it geometrically using pebbles).

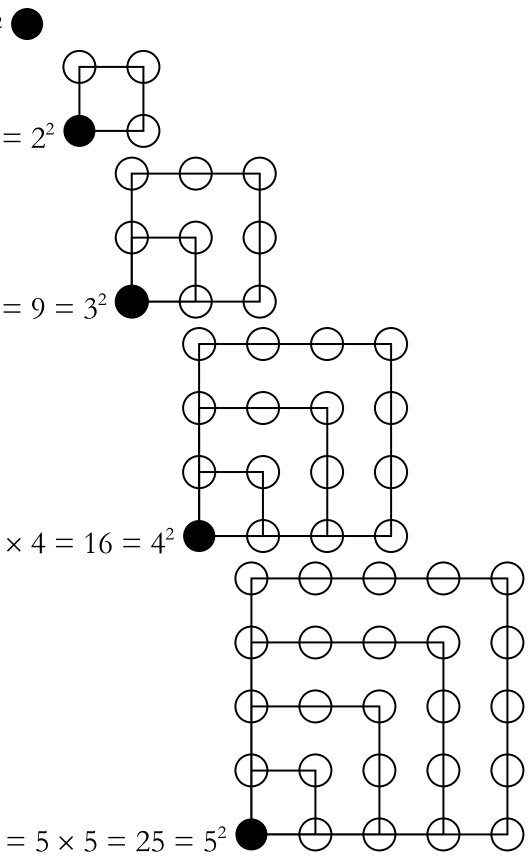
$1 = 1 \times 1 = 1 = 1^2$

$1 + 3 = 2 \times 2 = 4 = 2^2$

$1 + 3 + 5 = 3 \times 3 = 9 = 3^2$

$1 + 3 + 5 + 7 = 4 \times 4 = 16 = 4^2$

$1 + 3 + 5 + 7 + 9 = 5 \times 5 = 25 = 5^2$



Note that the next odd number is added to the previous sum by adding a *gnomon* of dots (similar in shape to a carpenter's square).

Source: iStockPhoto

## SEQUENCE OF CUBES

The concept of volume (three-dimensions) helps us understand the sequence of cubes. How many inch cubes are contained in a Rubik's cube of dimensions 3 by 3 by 3? To find the volume, we calculate $3^3 = 3 \times 3 \times 3 = 27$. The volume is 27 cubic inches or 27 $in^3$.

# POWER SEQUENCES

BY JAMES D. NICKEL

The sequence of cubes gives you progressively larger volumes of a cube as the length of one of its sides increases according to the sequence of positive integers: 1, 2, 3, 4, 5, …

| Positive Integers cubed | $1^3$ | $2^3$ | $3^3$ | $4^3$ | $5^3$ | … |
|---|---|---|---|---|---|---|
| Sequence of Cubes | 1 | 8 | 27 | 64 | 125 | … |

*Cubing a number* means raising that number to the power of 3. The inverse of cubing is "extracting the cube root" or "taking the cube root" or "finding the third root of a number." $\sqrt[3]{\ }$ is the symbol for this operation. In general, the symbol $\sqrt[n]{a}$ means "take the $n^{th}$ root of the number *a*."

Since $2^3 = 2 \times 2 \times 2 = 8$, then $\sqrt[3]{8} = 2$. Since $2 \times 2 \times 2 = 8$, the cube root of a positive integer is always positive. Now note: $(-2) \times (-2) \times (-2) = -8$. This means that the cube root of a negative integer is always negative; i.e., $\sqrt[3]{-8} = -2$

Let's search for a pattern in the positive cubic numbers. If a number is a perfect cube (quite rare in the cavalcade of positive integers), we can develop a method for extracting its cube root. Note carefully the pattern in the last digits.

| Number | Cube | Last Digit of Number | Last Digit of Cube | Number | Cube | Last Digit of Number | Last Digit of Cube |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 16 | 4096 | 6 | 6 |
| 2 | 8 | 2 | 8 | 17 | 4913 | 7 | 3 |
| 3 | 27 | 3 | 7 | 18 | 5832 | 8 | 2 |
| 4 | 64 | 4 | 4 | 19 | 6859 | 9 | 9 |
| 5 | 125 | 5 | 5 | 20 | 8000 | 0 | 0 |
| 6 | 216 | 6 | 6 | 21 | 9261 | 1 | 1 |
| 7 | 343 | 7 | 3 | 22 | 10,648 | 2 | 8 |
| 8 | 512 | 8 | 2 | 23 | 12,167 | 3 | 7 |
| 9 | 729 | 9 | 9 | 24 | 13,824 | 4 | 4 |
| 10 | 1000 | 0 | 0 | 25 | 15,625 | 5 | 5 |
| 11 | 1331 | 1 | 1 | 26 | 17,576 | 6 | 6 |
| 12 | 1728 | 2 | 8 | 27 | 19,683 | 7 | 3 |
| 13 | 2197 | 3 | 7 | 28 | 21,952 | 8 | 2 |
| 14 | 2744 | 4 | 4 | 29 | 24,389 | 9 | 9 |
| 15 | 3375 | 5 | 5 | 30 | 27,000 | 0 | 0 |

Going from number to cube is "cubing the number" and going from cube to number is "taking the cube root." The pattern is simple:

- If the last digit of a perfect cube is 0, 1, 4, 5, 6, or 9, the last digit of its cube root ends in the same digit.
- If the last digit of a perfect cube is 2, the last digit of its cube root is 8.
- If the last digit of a perfect cube is 8, the last digit of its cube root is 2.
- If the last digit of a perfect cube is 3, the last digit of its cube root is 7.
- If the last digit of a perfect cube is 7, the last digit of its cube root is 3.

Now note the table of the cubes of the multiples of 10.

| Multiple of 10 | Cube | Cube root |
|---|---|---|
| 10 | 1000 | $\sqrt[3]{1000} = 10$ |
| 20 | 8000 | $\sqrt[3]{8000} = 20$ |
| 30 | 27,000 | $\sqrt[3]{27,000} = 30$ |
| 40 | 64,000 | $\sqrt[3]{64,000} = 40$ |
| 50 | 125,000 | $\sqrt[3]{125,000} = 50$ |
| 60 | 216,000 | $\sqrt[3]{216,000} = 60$ |
| 70 | 343,000 | $\sqrt[3]{343,000} = 70$ |
| 80 | 512,000 | $\sqrt[3]{512,000} = 80$ |
| 90 | 729,000 | $\sqrt[3]{729,000} = 90$ |
| 100 | 1,000,000 | $\sqrt[3]{1,000,000} = 100$ |

If a number is less than 1,000,000, then its cube root is less than 100. If a number is less than 125,000, then its cube root is less than 50. If a number is less than 27,000, then its cube root is less than 30. If a number is between 27,000 and 125,000, then its cube root is between 30 and 50.

What is the cube root of 226,981? The cube root must be between 60 and 70. Since the last digit is 1, the cube root must be 61!

Let's try a few more. What is the cube root of:

- 970,299? It is between 90 and 100. Since the last digit is 9, $\sqrt[3]{970,299} = 99$
- 157,464? It is between 50 and 60. Since the last digit is 4, $\sqrt[3]{157,464} = 54$
- 389,017? It is between 70 and 80. Since the last digit is 7, $\sqrt[3]{389,017} = 73$
- 681,472? It is between 80 and 90. Since the last digit is 2, $\sqrt[3]{681,472} = 88$

## SOME AMAZING CONNECTIONS

Square numbers connect to cubic numbers in some wonderful ways. First, let's travel into the past. Archaeologists have discovered thousands of clay tablets (written with an instrument called a stylus) in the ancient ruins of Babylonia. Some of these tablets contain what appear to be mathematical tables (of course, the Babylonians did not use the base 10 decimal system). Translated into the base 10 decimal system, here is one table:

| x | y = f(x) | Δy | Δ(Δy) | Δ(Δ(Δy)) |
|---|---|---|---|---|
| 1 | 2 | | | |
| 2 | 12 | 10 | | |
| 3 | 36 | 24 | 14 | |
| 4 | 80 | 44 | 20 | 6 |
| 5 | 150 | 70 | 26 | 6 |
| 6 | 252 | 102 | 32 | 6 |
| 7 | 392 | 140 | 38 | 6 |
| 8 | 576 | 184 | 44 | 6 |

| x | y = f(x) | Δy | Δ(Δy) | Δ(Δ(Δy)) |
|---|---|---|---|---|
| 9 | 810 | 234 | 50 | 6 |
| 10 | 1100 | 290 | 56 | 6 |

I have added the last three columns. The tablets only have the first two. Notice the Δ (Greek letter "delta") symbol. In mathematics, this symbol means "difference."[1] Column three, Δy, in the table is the difference between *x* and *y* as *x* increases one unit. For example, 12 – 2 = 10, 36 – 12 = 24, etc. Column four, Δ(Δy), in the table is the difference between *y* and Δy as *x* increases one unit. For example, 24 – 10 = 14, 44 – 24 = 20, etc. Column four, Δ(Δ(Δy)), in the table is the difference between Δy and Δ(Δy) as *x* increases one unit. For example, 20 – 14 = 6, 26 – 29 = 6, etc. In the Δ(Δ(Δy)), the difference is *always the same*, or there is a "common difference." This means that the 4th column, Δ(Δy), is an arithmetic sequence. This observation is significant as we shall soon note.

Put on your mathematical hats and see if you can discover a pattern.

- 12 = 3 × 4
- 36 = 4 × 9
- 80 = 5 × 16

Is there a connection between y and x?

- When x = 2, $2^2 = 4$
- When x = 3, $3^2 = 9$
- When x = 4, $4^2 = 16$

We have the beginnings of a pattern! Note also:

- When x = 2, 3 = 2 + 1
- When x = 3, 4 = 3 + 1
- When x = 4, 5 = 4 + 1

More patterns! Let's put it all together:

- When x = 1, we have: $(1 + 1)(1^2) = (2)(1) = 2$
- When x = 2, we have: $(2 + 1)(2^2) = (3)(4) = 12$
- When x = 3, we have: $(3 + 1)(3^2) = (4)(9) = 36$
- When x = 4, we have: $(4 + 1)(4^2) = (5)(16) = 80$

In general, $y = f(x) = (x + 1)x^2 = x^3 + x^2$ (applying the Distributive Property). Hence, *the second column is a list of the sums of cubes and squares.*

- $2 = 1^3 + 1^2$
- $12 = 2^3 + 2^2$
- $36 = 3^3 + 3^2$
- $80 = 4^3 + 4^2$

How did the ancient Babylonians use this table? We are not sure, but they generated it for some reason.

[1] It is also a symbol denoted a triangle.

# Power Sequences

by James D. Nickel

Let's return to the $\Delta(\Delta(\Delta y))$ column where the difference is constant. This is the *third* difference. The highest exponent of the function $y = f(x) = x^3 + x^2$ is 3. Is there a connection? Yes! You will learn more about this in advanced algebra course.

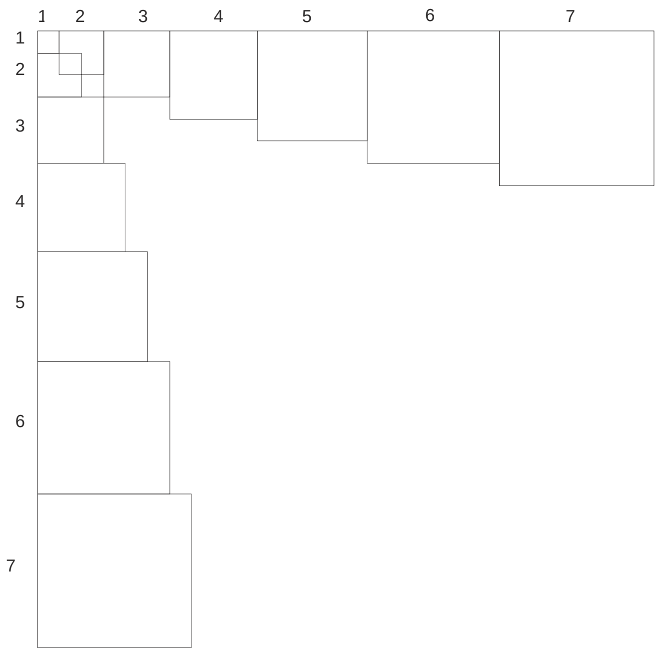



Figure 1

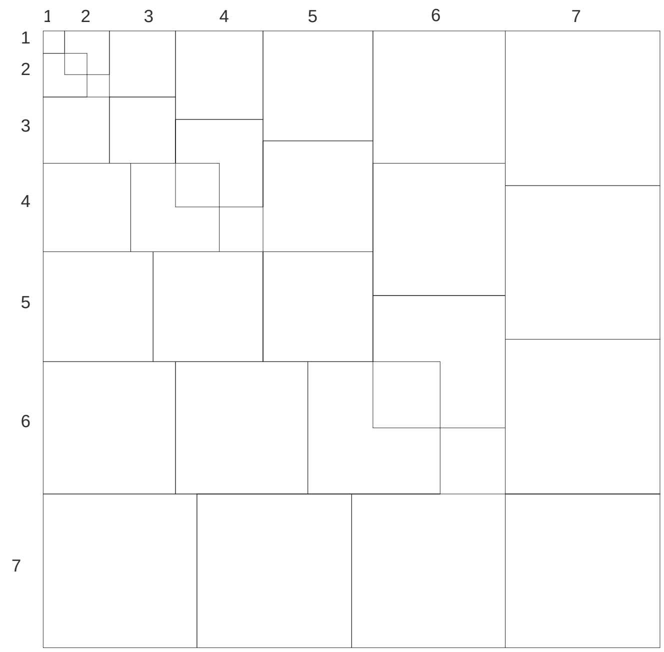



Figure 2

We conclude with one final connection between square numbers and cubic numbers. Since square numbers relate to area, let's consider the area of a square whose side measures 1 + 2 + 3 + 4 + 5 + 6 + 7 (an arithmetic sequence whose sum is 28. The area of this square is therefore $(1 + 2 + 3 + 4 + 5 + 6 + 7)^2$.

Note the squares created in Figure 1. We have 1 square of length 1 and 2 squares of length 2. Let's continue this process. In Figure 2, we have generated 3 squares of length 3, 4 squares of length 4, etc., to 7 squares of length 7. Take some time "to see" this in the figure.

In Figure 3, note the 3 gnomons (colored "reversed L" shaped figures). The brown square has side 1. The red gnomon consists of 3 squares with sides 3 in L shape. The blue gnomon consists of 5 squares with sides 5 in L shape. The green gnomon consists of 7 squares with sides 7 in L shape.

The colored gnomons make up part of the area of the large square. What remains?

- 2 separate squares with sides 2.
- 4 squares with sides 4 (2 in each of 2 separate strips).
- 6 squares with sides 6 (3 in each of 2 separate strips).

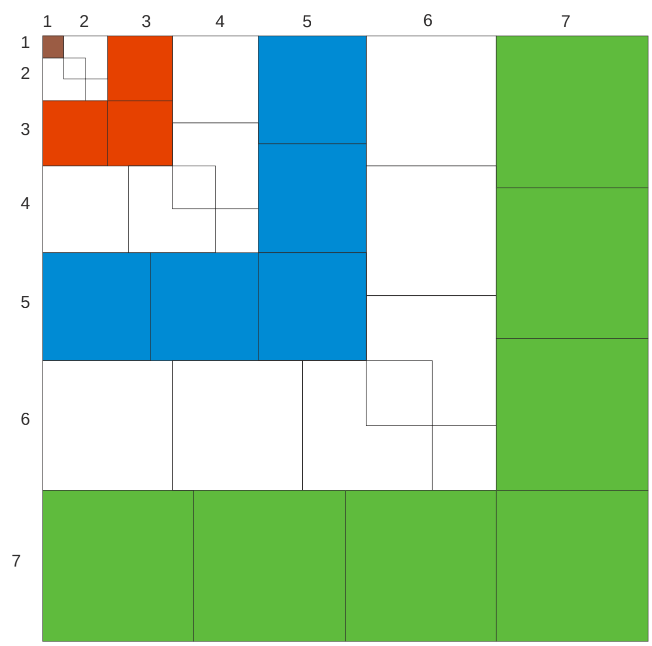



Figure 3

Each region of overlap corresponds to a hole of the same size. The area of the large square can be stated as follows:

- The area of 1 square with side 1: $1 \times 1^2 = 1^3$
- The area of 2 squares with sides 2: $2 \times 2^2 = 2^3$
- The area of 3 squares with sides 3: $3 \times 3^2 = 3^3$
- The area of 4 squares with sides 4: $4 \times 4^2 = 4^3$
- The area of 5 squares with sides 5: $5 \times 5^2 = 5^3$
- The area of 6 squares with sides 6: $6 \times 6^2 = 6^3$
- The area of 7 squares with sides 7: $7 \times 7^2 = 7^3$

Hence, $(1 + 2 + 3 + 4 + 5 + 6 + 7)^2 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$. Note, $(1 + 2 + 3 + 4 + 5 + 6 + 7)^2 = 28^2 = 784$ and $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = 1 + 8 + 27 + 64 + 125 + 216 + 343 = 784$. What a marvelous way to end our discussion of power sequences!