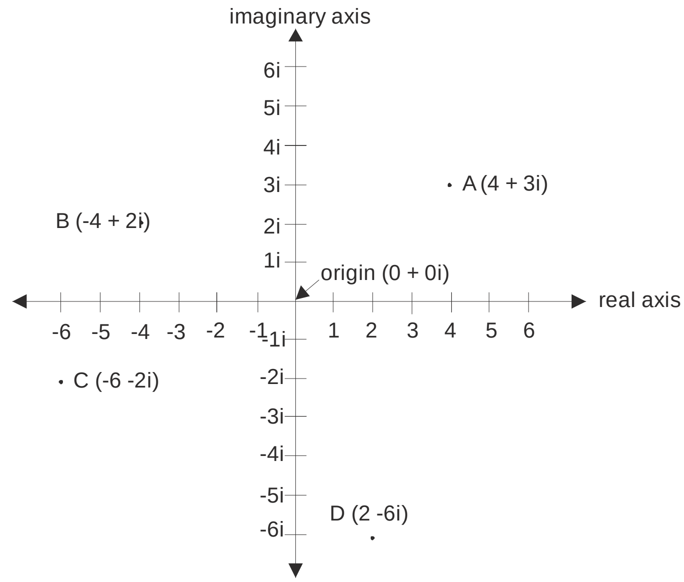
Dr. Julia, meet Dr. Mandelbrot

by James D. Nickel

Early in the 20th century, the French mathematician Gaston Maurice Julia (1893-1978), after losing his nose fighting in World War I, devised an iterative mathematical formula, using the arithmetic of complex numbers, for what is now called the *Julia Set*. Recall that a complex number is of the form a + bi where a, b $\in \mathbb{R}$ and i = $\sqrt{-1}$ and that every complex number can represented in what is called a complex number plane.

> Note: This essay is extracted from a Lesson from the forthcoming textbook *Mathematics: Building on Foundations.*



Complex Number Plane

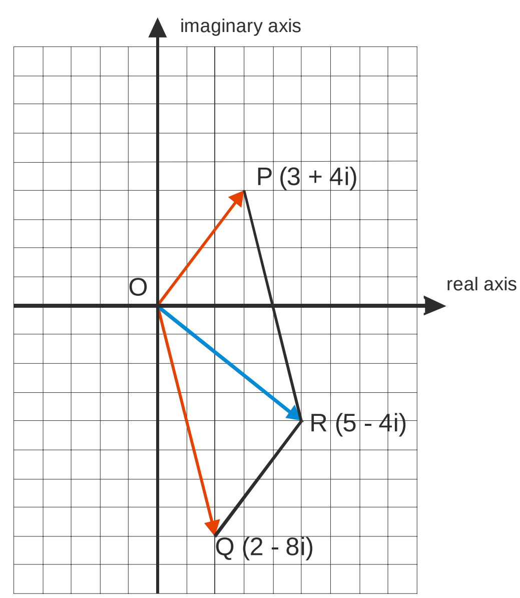
There is an arithmetic to complex numbers that is fascinating. For example, adding two complex numbers is just a matter of adding their corresponding real number parts and their corresponding imaginary number parts. In general, (a + bi) + (c + di) = (a + c) + (b + d)i. For example, (3 + 4i) + (2 – 8i) = (5 – 4i). Scientists, upon seeing the graphical representation of the addition of complex numbers, immediately understood the complex number sum as representing the resultant vector of two independent forces (based upon Isaac Newton's parallelogram law of addition of forces).

Multiplying two complex numbers works out like this:

$(a + bi)(c + di) = ac + adi + bci + bdi^2 =$
$ac + i(ad + bc) + bdi^2$

Since, by definition, $i^2 = -1$, then we get:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$



Vector Addition

Julia's formula is a simple iterative scheme. Given a complex number of the form a + bi, this number is first squared. In general, $(a + bi)^2 = (a + bi)(a + bi) = a^2 + abi + abi + b^2i^2 = a^2 + 2abi + b^2i^2$ Again, since $i^2 = -1$, then we get:

$$(a + bi)^2 = a^2 - b^2 + 2abi$$

Once this number is squared, another complex number of the form c + di is added to it. In general, we get:

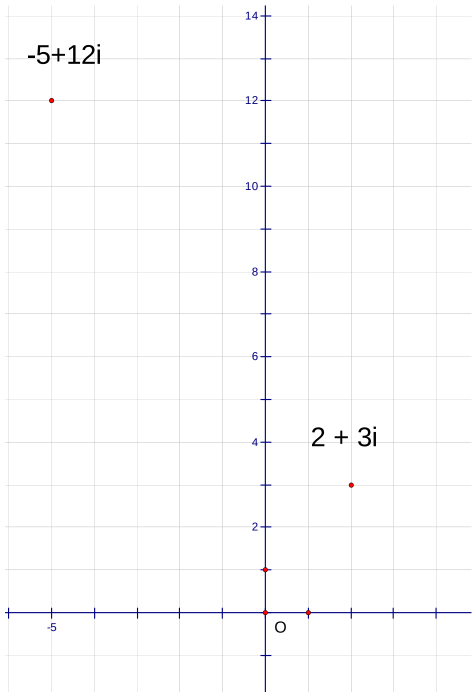
$$(a + bi)^2 + c + di = (a^2 - b^2) + c + (2ab + d)i$$

This new number is now input back into the iterative process; i.e., it is squared and c + di is added again to this square. The result is squared, c + di is added to it, and the process continues *ad infinitum*.

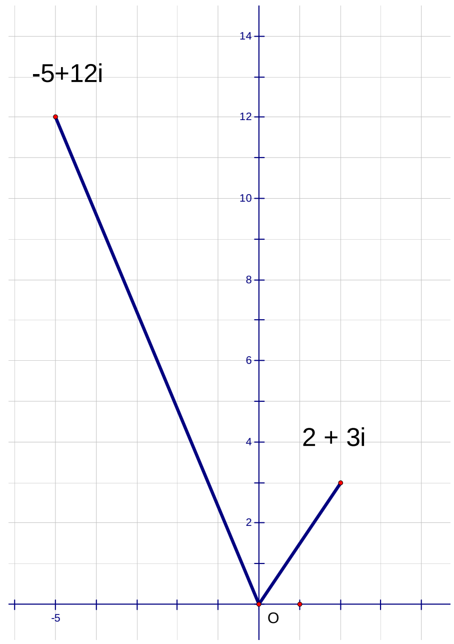
Pause the play button and wait nearly seven decades until the invention of the personal computer. With the processing power of these machines, along with their enhanced graphical display, Julia's iterations could enter the world of the visual. Wonder of wonders, the graphs of *Julia sets* boggle the mind in both intricacy and beauty.

Before we visualize these "complex" wonders (pun intended), let's simplify Julia's iteration formula. Given a complex number of the form a + bi, we square it. Then, we square the result. We continue this process *ad infinitum*. What happens when we picture this simpler iteration on the complex number plane?

Let's start with an example. Consider the complex number 2 + 3i. $(2 + 3i)^2 = -5 + 12i$. Let's plot these points and draw some conclusions. After plotting these points on the complex number plane, let's first draw a line segment from the origin to each.

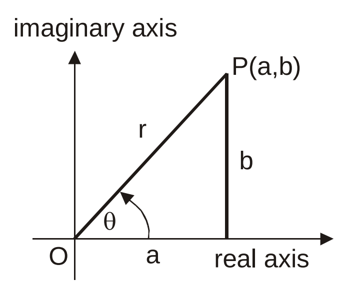


$(2 + 3i)^2 = 5 + 12i$



$(2 + 3i)^2 = 5 + 12i$

Let's now make use of some trigonometry to quantify what we are seeing. We can also represent complex numbers geometrically in polar form. In the figure, the phase angle θ (a physics term) that the line segment $\overline{OP}$ makes with the positive real axis is called the *argument*[1] or *amplitude*[2] of the complex number a + bi. The length *r* or $\overline{OP}$ is called the *absolute value* or *modulus*[3] of a + bi. By the Pythagorean Theorem:



$$r = \sqrt{a^2 + b^2}$$

It was the German mathematician Carl Friedrich Gauss (1777-1855) who introduced the vector (direction number) concept to the complex number plane. Since a + bi can be considered a vector, |a + bi| is the magnitude of the vector and is defined as follows:

$$|a + bi| = \sqrt{a^2 + b^2}$$

We can note four more useful relationships from the figure. We see that:

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[1] Argument is Latin for "make clear."
[2] Amplitude is Latin for "large."
[3] Modulus is Latin for "measure."

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1. $\cos\theta = \dfrac{a}{r}$. Hence, $a = r\cos\theta$.
2. $\sin\theta = \dfrac{b}{r}$. Hence, $b = r\sin\theta$.
3. $\tan\theta = \dfrac{b}{a} \Leftrightarrow \theta = \tan^{-1}\left(\dfrac{b}{a}\right)$ – on your calculator find the quotient of b/a and then hit the $\tan^{-1}$ key (arctan or the inverse of the tangent function).
4. $r = \sqrt{a^2 + b^2}$

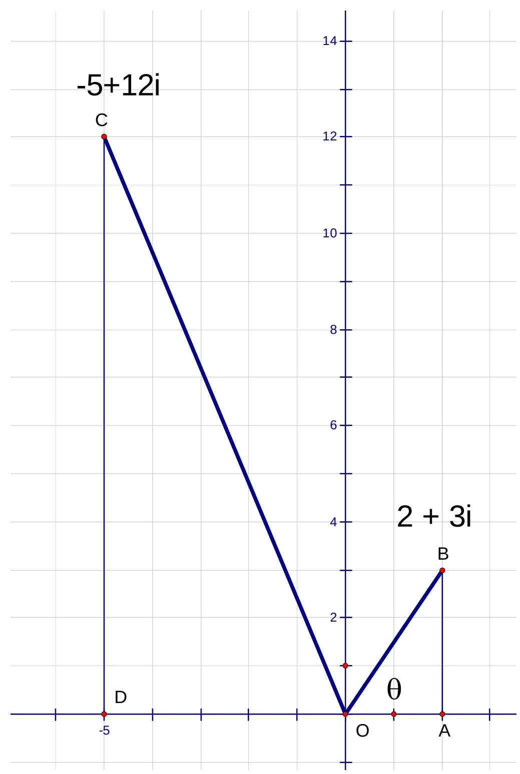
Therefore, $a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$. The expression $r(\cos\theta + i\sin\theta)$ is commonly abbreviated as r cis θ where c represents *cos*, s represents *sin*, and *i* represents $\sqrt{-1}$. r cis θ is called the *polar* or *trigonometric form* of a complex number.

Let's now determine r and θ for the complex number 2 + 3i. We get:

$$r = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$$
$$\theta = \tan^{-1}\left(\frac{3}{2}\right) = \tan^{-1}(1.5) = 56.31^\circ$$

Next, let's determine r_1 and θ_1 for the complex number $(2 + 3i)^2 = -5 + 12i$.

$$r_1 = \sqrt{(-5)^2 + 12^2} = \sqrt{25+144} = \sqrt{169} = 13$$
$$\theta_1 = \tan^{-1}\left(-\frac{12}{5}\right) = \tan^{-1}(-2.4) = -67.38^\circ$$



(2 + 3i)² = 5 + 12i

What conclusions can we draw? The first observation we can make is that $r_1 = r^2$ since $13 = \sqrt{13}^2$. In other words, the magnitude *r* of the vector is *squared*. How do we compare θ and θ_1? For 2 + 3i, $\theta = \angle BOA$. For 5 + 12i, $\theta_1 = \angle COD$. Recall that, from our knowledge of trigonometry $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ will always be negative if the point or ordered pair lies in the second quadrant.

Let's now consider the value of θ_1. The measurement we have calculated θ_1 starts at the left half of the real number axis (x-axis) and moves clockwise to $\overline{OC}$. We are seeking the measure of $\angle AOC$. To get this measure, we start at the right half of the real number axis and move counterclockwise until we reach $\overline{OC}$. We know that the degree measure from the right half of the real axis to the top half of the imaginary axis (y-axis) is 90°. From the top half of the imaginary axis to $\overline{OC}$ is 90° – 67.38° = 22.62°. Hence, $\angle AOC$ = 90° + 22.62° = 112.62°. Compare this measure with $\theta = \angle BOA = 56.31^\circ$. $\theta_1 = 2\theta$! θ, when measured counter-clockwise from the real axis, *doubles!*

Let's summarize our conclusions. When we square a complex number a + bi, then (1) the magnitude *r* is squared and (2) θ is doubled. Quite amazing, isn't it?

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Now, let's return to Julia sets and let's try to understand what is meant by the term. Applying the simplified iterative schemed (square the complex number each time) to 2 + 3i, we get:

$(2 + 3i)^2$	=	-5 + 12i
$(-5 + 12i)^2$	=	-119 – 120i
$(-119 – 120i)^2$	=	-239 + 28,560i
$(-239 + 28{,}560i)^2$	=	-815,616,479 –13,651,680i
$(-815{,}616{,}479 –13{,}651{,}680i)^2$	=	665,043,872,449,535,041+ 22,269,070,348,069,440i

My, what large "components" these successive complex numbers are getting close to! If we try to plot them (we won't) we should realize that the points are galloping far, far away *from* the origin.

Next, let's try a different complex number, 0.3 + 0.6i. We get:

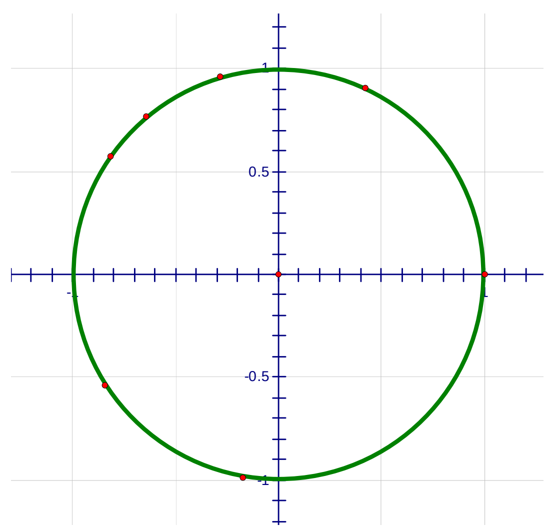
$(0.3 + 0.6i)^2$	=	-0.27 + 0.36i
$(-0.27 + 0.36i)^2$	=	-0.0567 – 0.1944i
$(-0.0567 – 0.1944i)^2$	=	-0.03457647 + 0.02204496i
$(-0.03457647 + 0.02204496i)^2$	=	0.0007095520163 – 0.001524473796i
$(0.0007095520163 – 0.001524473796i)^2$	=	-0.000001820556291 – 0.000002163386912i

In contrast, what small "components" these successive complex numbers are getting close to! If we try to plot them (again, we won't) we should realize that the points are squeezing closer and closer *toward* the origin.

One more: Let's try the same iterative scheme for the complex number 0.6 + 0.8i. We get:

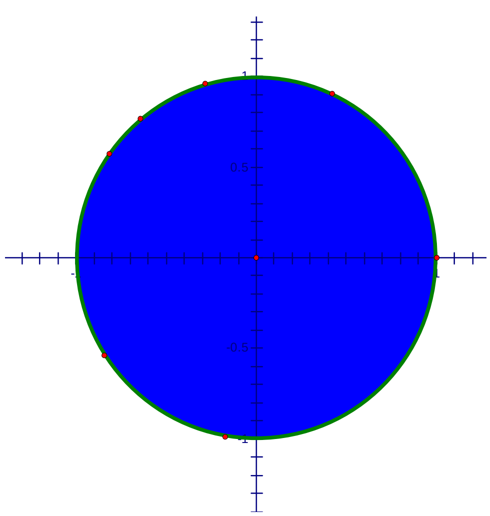
$(0.6 + 0.8i)^2$	=	-0.28 + 0.96i
$(-0.28 + 0.96i)^2$	=	-0.8432 – 0.5376i
$(-0.8432 – 0.5376i)^2$	=	0.4219724 + 0.90660864i
$(0.4219724 + 0.90660864i)^2$	=	-0.6438784522 + 0.7651277924i
$(-0.6438784522 + 0.7651277924i)^2$	=	-0.1708410775 – 0.9852985975i

What is happening in this instance? Let's plot these points. Amazingly, they all lie on the circumference of a circle of radius = 1! Can we analytically explain why this is so? Our starting point is 0.6 + 0.8i. In this case, a = 0.6 and b = 0.8. Hence, $r=\sqrt{(0.6)^2+(0.8)^2}=\sqrt{0.36+0.64}=\sqrt{1}=1$. Hence, when we square the complex number 0.6 + 0.8i, r will be $1^2 = 1$. Hence, for successive squares of 0.6 + 0.8i, r will *always* be 1.



Starting point: 0.6 + 0.8i

Let's take our three starting points, 2 + 3i, 0.3 + 0.6i, and 0.6 + 0.8i, and see if we can come to some conclusions. If a starting point is *outside* the circle centered at the origin with radius 1 (e.g., 2 + 3i), then the iteration values gallop off to infinity. If a starting point is *inside* the circle centered at the origin with radius 1 (e.g., 0.3 + 0.6i), then the iteration values squeeze toward


Elementary Julia Set

the origin. Finally, if a starting point is *on* the circle centered at the origin with radius 1(e.g., 0.6 + 0.8i), then the iteration values, when plotted, remain on that circle.

Hence, the circle acts as a *boundary* separating *all* the starting points that lie outside it (and iteratively, gallop off to infinity) from *all* the starting points that lie inside it (and iteratively, squeeze toward the circle's origin). This circle, the interface between these two sets of numbers, is what we mean by a rudimentary *Julia Set*. We can picture the Julia Set by filling in the circle. In our figure, the space colored blue represents the set of starting points whose iterations *do* not gallop off to infinity.

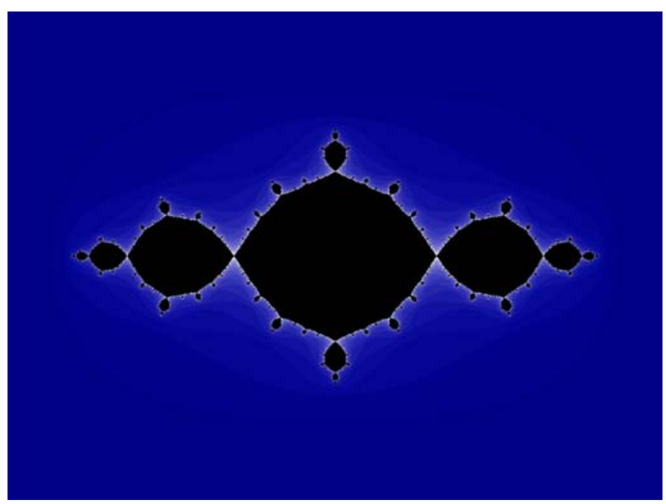
The circle in the complex number plane is the first and basic Julia Set. Now, we are ready to make things interesting and therefore astonishingly beautiful. To do so, we add one extra ingredient to our iterative process. We start with a "seed" complex number, a + bi. This number always remains fixed. For example, we let a + bi = -1 + 0i. We then choose any complex number from among an infinite number of possibilities. Let's choose 0 + 0i. We square 0 + 0i, add -1 + 0i to it. Then, we take that answer, square it, and add -1 + 0i to it. We continue this iterative process *ad infinitum*. Here is what we get:

$(0 + 0i)^2 + (-1 + 0i)$	=	-1 + 0i
$(-1 + 0i)^2 + (-1 + 0i)$	=	0 + 0i
$(0 + 0i)^2 + (-1 + 0i)$	=	-1 + 0i
$(-1 + 0i)^2 + (-1 + 0i)$	=	0 + 0i
⋮	=	⋮

Let's choose another complex number: 2 + 3i. We get:

$(2 + 3i)^2 + (-1 + 0i)$	=	-6 + 12i
$(-6 + 12i)^2 + (-1 + 0i)$	=	-109 – 144i
$(-109 - 144i)^2 + (-1 + 0i)$	=	-8856 + 31,392i
$(-8856 + 31{,}392i)^2 + (-1 + 0i)$	=	-907,028,929 – 556,015,104i
⋮	=	⋮

This "extra ingredient", adding (-1 + 0i) each time, generates a Julia set just as before. Certain starting points will iteratively gallop off to infinity. Other starting points will *not* join this infinity race course. If we think of all points in the complex number plane as valid starting points, then some will iteratively race to infinity and others will not. The interface, the boundary, between these two sets of complex numbers is called the *Julia Set*. The Julia Set for the "seed" complex number of (-1 + 0i) is pictured; the computations were performed by a modern personal computer.[4]

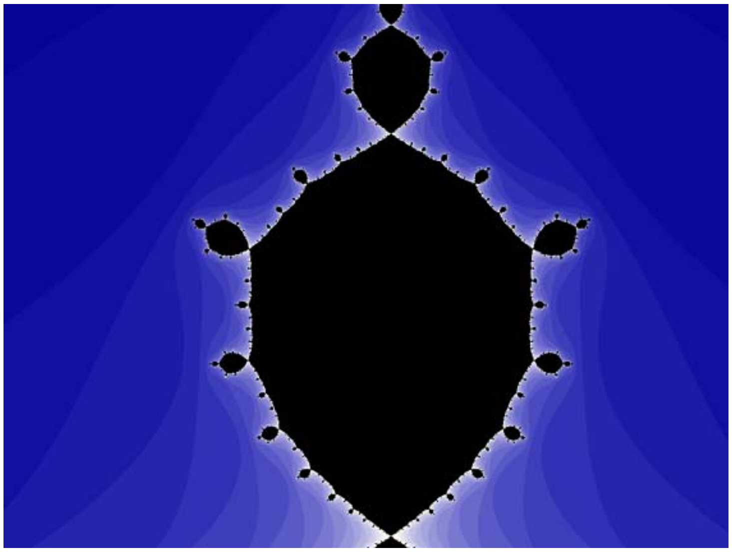


Julia Set, seed = (-1 + 0i)

All the complex number points in the dark region do *not* gallop off to infinity. Those in the blue region do. The blue region that is darker indicates that the iterative process "gallops" faster. Hence, shade of color represents the pace of the "race to infinity." Please pause to gaze at this exquisite beauty. Words cannot express it properly. The wonder of Julia sets is that they *infinitely* self-replicate. Hence, you can "zoom" in at any point are recognize the same geometric patterns as the

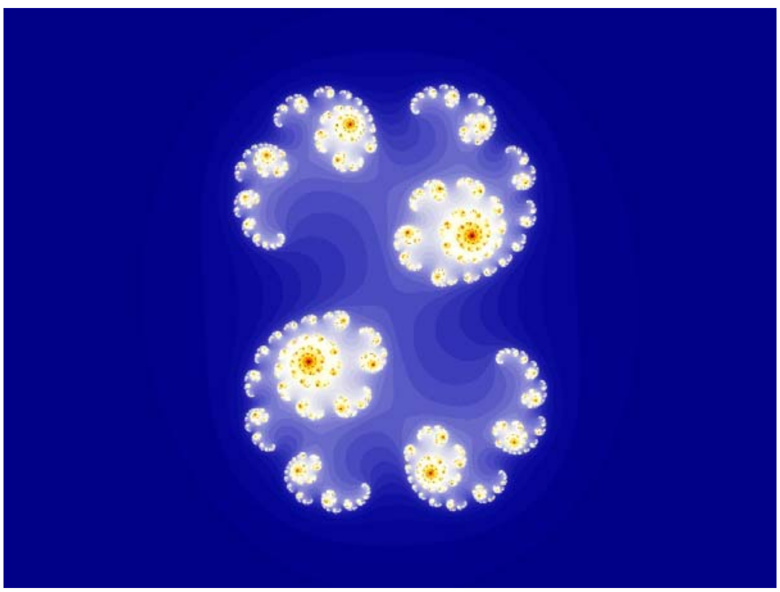
[4] I am using Ultra Fractal v4.04 (standard edition) to generate these fractal images. You can order this software from www.ultrafractal.com and experiment to your heart's content.

original. Here is one example of a "zoom" (we aimed our zoom at the top small "bubble"):

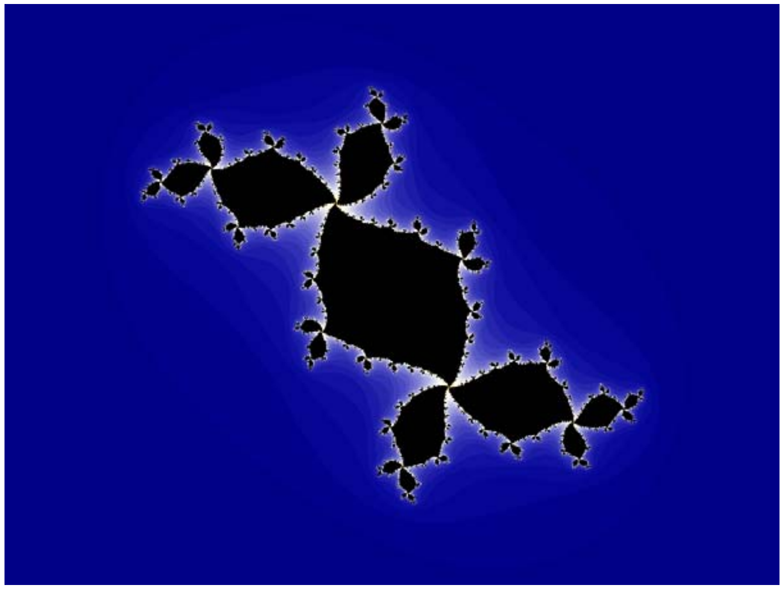


Julia Set, seed = (-1 + 0i) with "zoom" feature

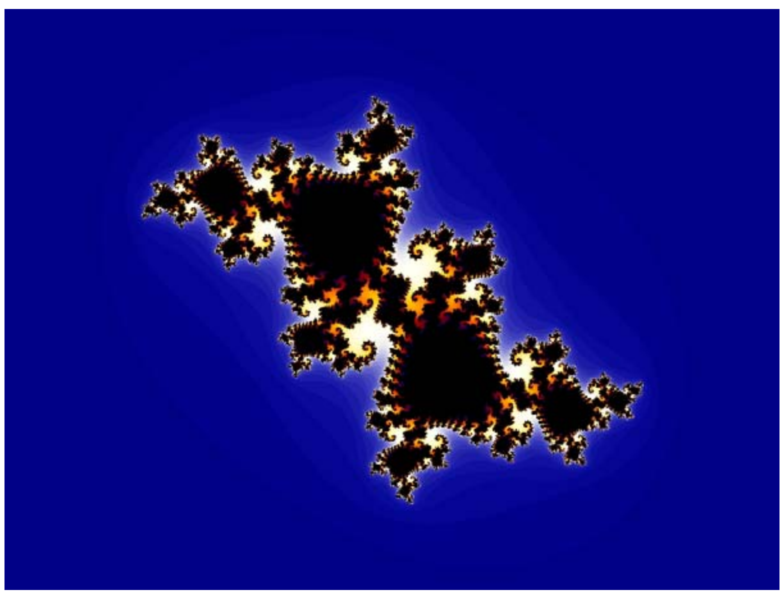
The next set of images will reveal the complex, intricate, and beautiful wonders of Julia sets:



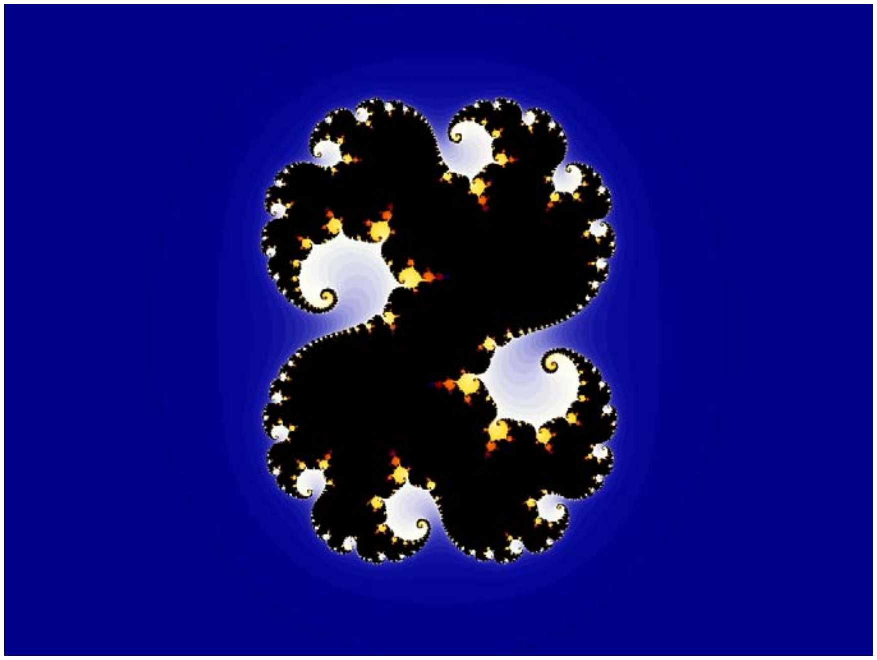
Julia Set, seed = (0.35 + 0.5i)



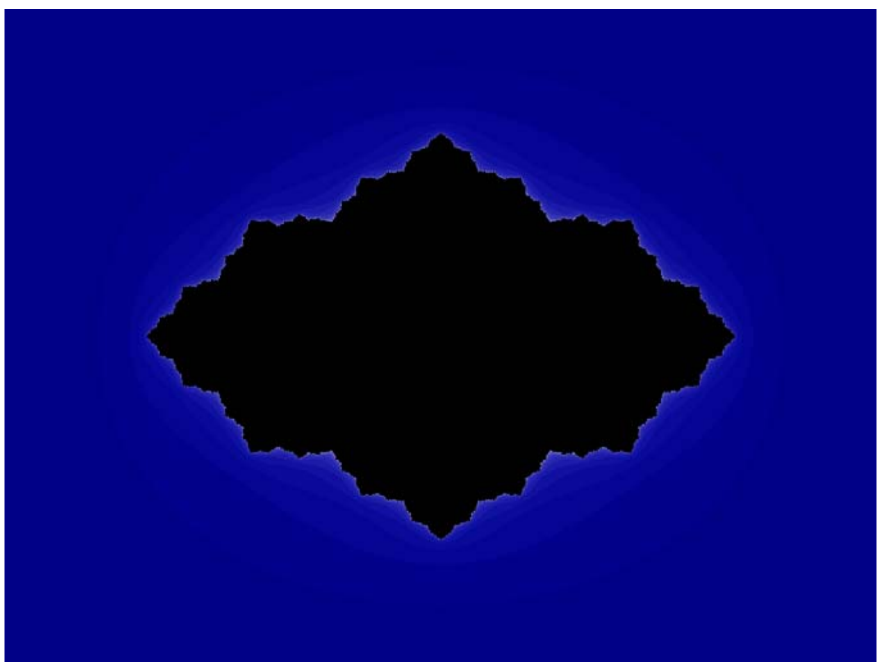
Julia Set, seed = (-0.15 + 0.75i)



Julia Set, seed = (-0.194 + 0.65i)



Julia Set, seed = (0.27334 + 0.00742i)



Julia Set, seed = (-0.475 + 0i)

Dr. Julia, meet Dr. Mandelbrot

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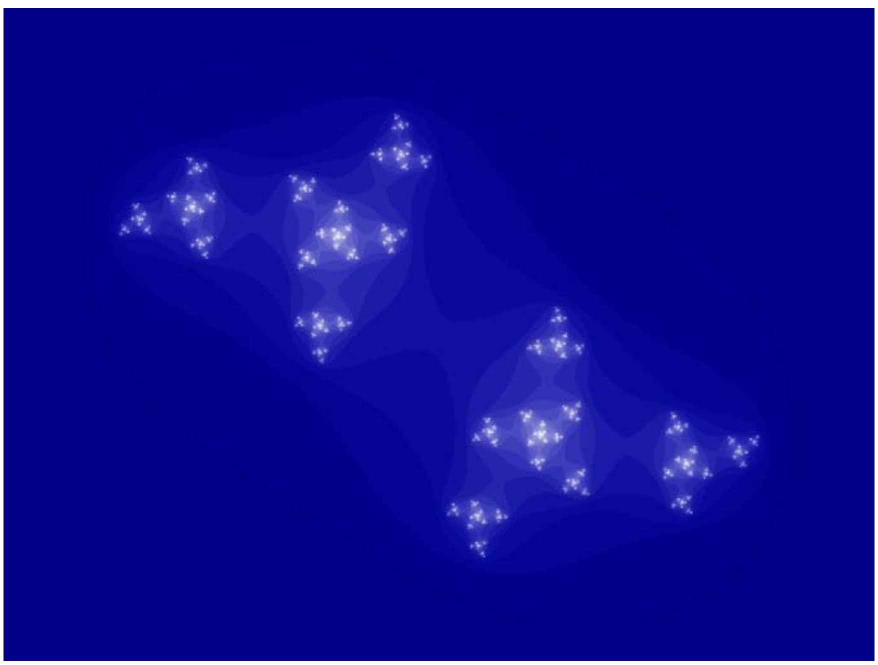
The number of Julia sets (and their associated graphical representation) is infinite because there is an infinite number of complex numbers. Hence, these intricate and infinitely replicated designs have no end. Mathematics and dazzling and spectacular beauty are *indeed* connected! You could immerse yourself in this immensity of beauty *forever!* Think about that!

Dr. Julia, now meet Dr. Mandelbrot. Benoît B. Mandelbrot (1924-), a Jewish-American mathematician, is known as the "father of fractal geometry." Mandelbrot resurrected Julia's work to new heights but standing back from these infinite sets and looking at the "big picture." Professor Mandelbrot endeavored to look at all of the (a + bi) Julia sets together. The set of all Julia sets is called the Mandelbrot Set. If you can believe it, the Mandelbrot Set is a mathematical object that captures all the information about the infinite number of Julia sets. For each point in the complex number plane, we can draw a unique (a + bi) Julia Set. The filled in Julia Set will either be one piece or more than one piece. Here is an example of a Julia Set that is more than one piece.



Julia Set, seed = (-0.475 + 0.88i)

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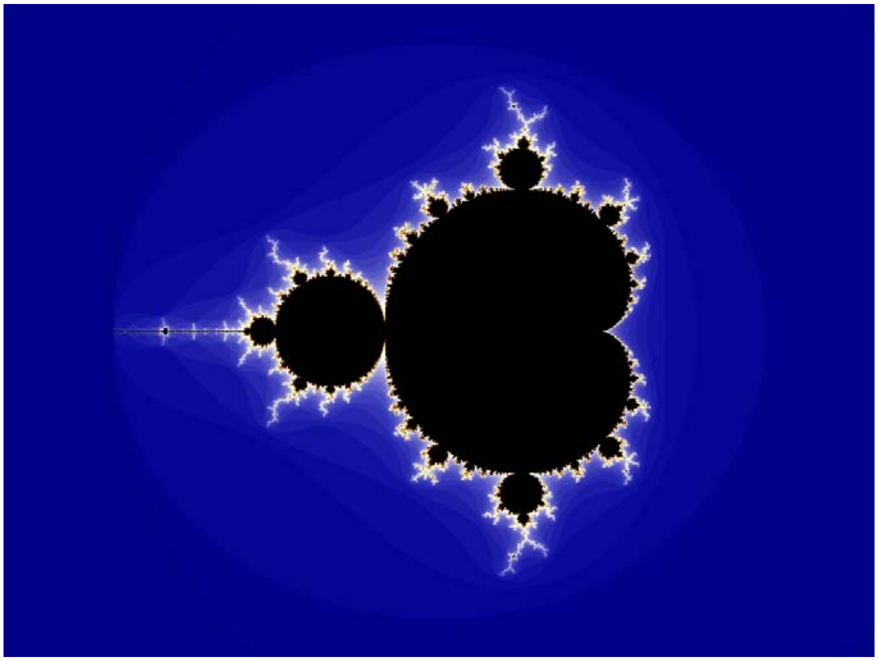
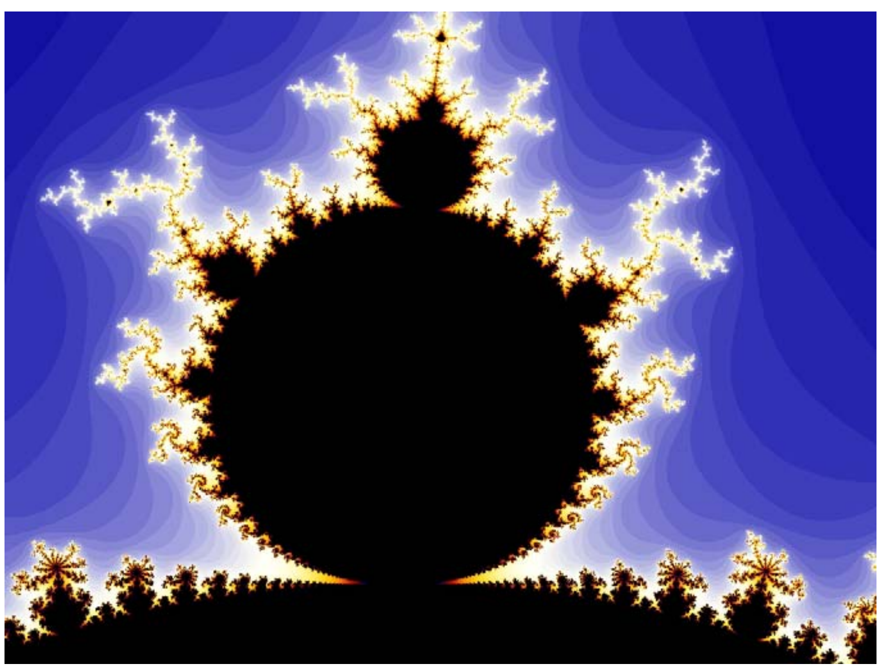
If the filled in (a + bi) Julia Set is connected, then we can denote that the point a + bi is a member of the Mandelbrot set. Hence, the Mandelbrot set is the set of all complex numbers a + bi exhibiting the property that the filled in (a + bi) Julia set *is just one piece*. The Mandelbrot set can be pictured. Here it is:



Mandelbrot Set

For each point in or near the Mandelbrot Set, we can associate a filled-in Julia Set that is in one piece. If the point is outside the Mandelbrot Set, then the associated filled-in Julia Set is made of multiple pieces. We can also "zoom" into different regions of the Mandelbrot Set in order to marvel at its infinitely detailed and intricate beauty. This zoom investigation can be repeated *ad infinitum*.



Mandelbrot Set, magnified top portion

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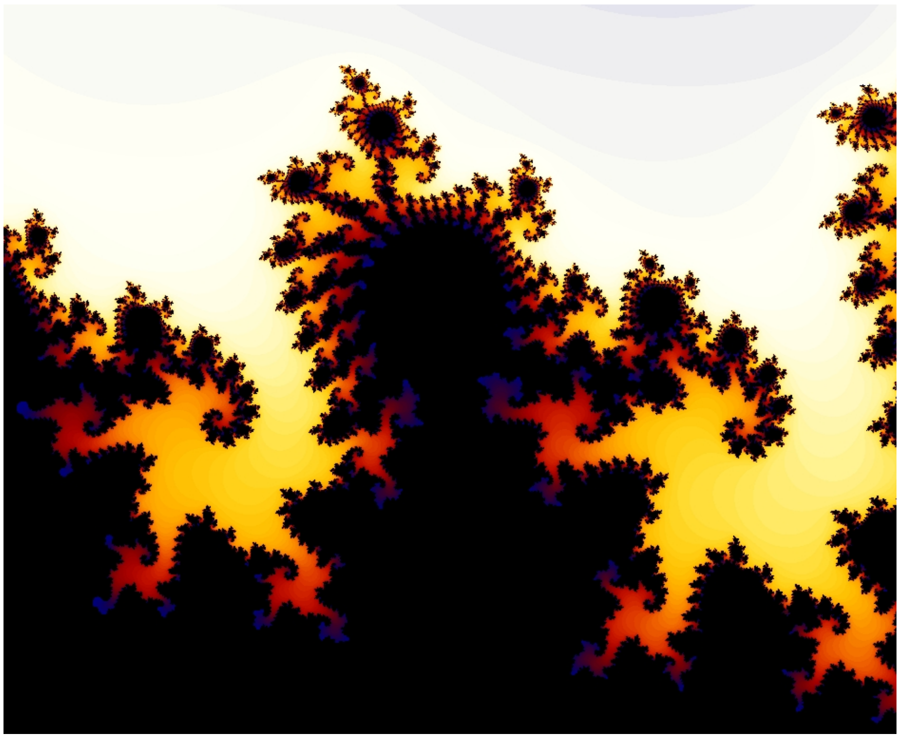
I hope you thoroughly enjoyed meeting Dr. Julia and observing the encounter between Dr. Julia and Dr. Mandelbrot. After this excursion into the mathematics of complex numbers, infinite iterations, Julia Sets, and the Mandelbrot Set, I hope that you have come to appreciate the reality that mathematics is *indeed* beautiful, not just in the logic of its many and elegant proofs, but in the infinite array of its sometimes fascinating and enthralling geometry. Praise God for the visual delight of this breathtaking splendor.



Mandelbrot Set, three magnifications deep