FRACTIONS A PLENTY

BY JAMES D. NICKEL

> Note: This essay is extracted from a Lesson from the forthcoming textbook *Mathematics: Building on Foundations*.

Let's consider this problem. It takes Mr. Swift $\frac{1}{2}$ of an hour (30 minutes) to run around 2 blocks at home while at the local YMCA it takes him $\frac{1}{3}$ of an hour (20 minutes) to run the same distance (there are no red lights at the local Y). What is the average time it takes him to run these distances? Average, in mathematics, is a useful means of measure and we will be making much use of it in subsequent lessons. For now, let's define this term along with two other closely related terms.

Another word for average is *arithmetic mean*.[1] The arithmetic mean of a group of *n* is defined as the sum of the numbers divided by *n*:

$$\bar{a} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

If I score 15, 20, 30, 20, and 25 points in 5 basketball games, then my average (arithmetic mean) is:

$$\frac{15+20+30+20+25}{5} = \frac{110}{5} = 22$$

The *mode*[2] of a group of numbers is the number that occurs most frequently in that group. Consider the following set of numbers: {0, 1, 1, 2, 2, 2, 3, 3, 3, 3, 5, 5, 6, 6, 6}. The number occurring most in this set is 3. It occurs 4 times.

The *median*[3] of a group of *n* numbers is the number such that just as many numbers are greater than it as are less than it. For example, in the set of numbers {1, 2, 3}, the median is 2. The median of {1, 1, 1, 2, 10, 15, 16, 20, 100, 105, 110} is 15. You can find the median from a list of numbers as follows. First, place the list in numerical sequence (from smallest to largest). Count the number of elements in the list. If the count is odd, then the median is the element in the exact middle. In our first two examples, the count was odd (3 for the first list and 11 for the second list). If the count is even, then the median is the arithmetic mean of the two numbers closest to the middle. For example, consider this set: {3, 4, 6, 8, 10, 15, 22, 28}. There are 8 elements. The middle elements are 8 and 10. The arithmetic mean of 8 and 10 is $\frac{8+10}{2} = 9$. Therefore the median of this set is 9.

The arithmetic mean, mode, and median are foundational concepts for a branch of mathematics called *statistics*.[4] Statisticians collect, organize, and interpret large groups of numerical data by means of sampling a subset of the given group.

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[1] Mean comes from the Latin word *medius* meaning "middle."

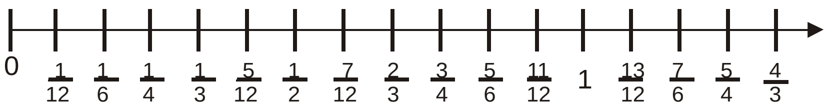
[2] Mode comes from the Latin word *modus* meaning "manner or fashion." Fashion implies that most people wear an item or use an item. It is fashionable to eat with a fork, knife, and spoon.

[3] Median is also derived from the Latin word *medius* meaning "middle." In this case, it is the "real" middle, not the "average" middle.

[4] Statistics has an interesting etymology. The German, Latin, and Italian words all mean "state or governmental affairs." This makes sense because the origin of this branch of mathematics resides in the desire of governmental officials to analyze certain characteristics of the citizenry of their political state. I bet the focus of this desire was the generation of tax revenues!

Now back to our problem. What is the arithmetic mean of $\frac{1}{2}$ and $\frac{1}{3}$? We sum the two numbers and divide by 2. Here goes: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}\div 2=\frac{5}{6}\times\frac{1}{2}=\frac{5}{12}$. $\frac{5}{12}$ of an hour (25 minutes) is the arithmetic mean.

We can always find a number between two given numbers by taking the arithmetic mean of the two numbers. Ponder carefully on that sentence. While you are thinking about it, let's construct our number line as follows:



Note that each fraction is represented in its lowest equivalent fraction, namely its *lowest term*. Notice how compact the number line is getting. Imagine what it would look like if we added fifths, sevenths, eighths, tenths, elevenths, thirteenths, etc. The fractions really start crowding each other then.

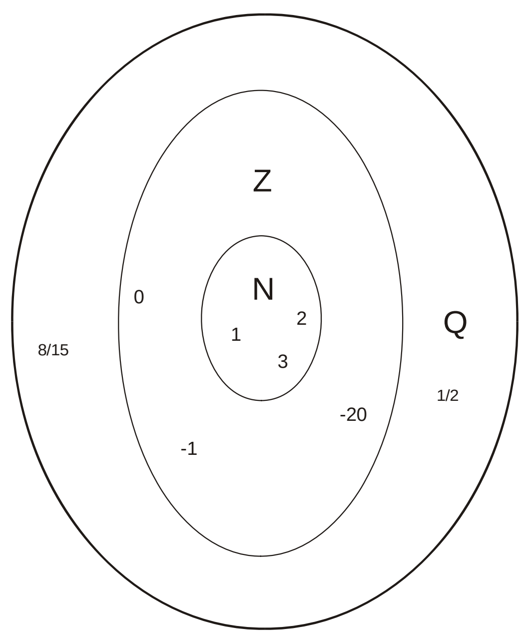
What we must notice also is that the whole numbers are part of this number line. We can consider whole numbers fractions because we can clothe them in "fractional dresses"; i.e., $1=\frac{1}{1}=\frac{2}{2}$, etc., $6=\frac{6}{1}=\frac{12}{2}$, etc. The whole numbers (in fact, the complete set of integers) and the fractions together make a set of numbers called *rational numbers* (denoted as $\mathbb{Q}$[5]). The word rational carries two meanings. In mathematics, a rational number is capable of being expressed as a quotient (or *ratio*[6]) of two integers. Here is a symbolic definition of a rational number where the symbol *iff* is shorthand for "if and only if."

Let a, b $\in \mathbb{Z}$ and b $\neq 0$. $\mathbb{Q}$ is a rational number *iff* $\mathbb{Q} = \frac{a}{b}$ and b $\neq 0$.

The second meaning carried by the word rational is the meaning of "being reasonable or able to consider or to reckon." By implication, the word rational implies that there are numbers that exist in a less than "rational" way. These "non-rational" numbers are called irrational numbers.

Before we inspect rational numbers in more detail, let's take a look at our number sets, the set of natural numbers, the set of whole numbers, the set of integers, and the set of rational numbers:

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$
$W = \{0, 1, 2, 3, 4, 5, \ldots\}$
$\mathbb{Z} = \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$
$\mathbb{Q} = \{\frac{a}{b}$ where a, b$\in \mathbb{Z}$ and b $\neq 0\}$



[5] $\mathbb{Q}$ comes from "quotient."

[6] Ratio, from the Latin means to "reckon or consider."

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Like a set of Russian nested dolls, $\mathbb{N}$ is contained in W, W is contained in $\mathbb{Z}$, and $\mathbb{Z}$ is contained in $\mathbb{Q}$. Using subset symbols, $\mathbb{N} \subset W$, $W \subset \mathbb{Z}$, and $\mathbb{Z} \subset \mathbb{Q}$. Each succeeding set builds upon the previous set.

Source: iStockPhoto

Let's return to the set of rational numbers and ask some questions. Can we enumerate all of them? Which rational number is closest to 0? Is there a *largest positive* rational number?

To simplify our analysis, let's just consider the rational numbers between 0 and 1. Since 0 is a rational number $\left(\frac{0}{1} = \frac{0}{2} = \frac{0}{3}, \text{etc.}\right)$, then let's pose this question. What is the smallest positive rational number greater than 0?

In our number line, it is obvious that the answer is not $\frac{1}{12}$. $\frac{1}{13}$ is smaller than $\frac{1}{12}$. But is $\frac{1}{13}$ the smallest? What about $\frac{1}{100}$? Now, that's small. But, $\frac{1}{101}$ is smaller than $\frac{1}{100}$. Okay, you say, let's try $\frac{1}{1,000,000}$. Beat that! $\frac{1}{1,000,001}$ is smaller than $\frac{1}{1,000,000}$. We have seen this type of argument before. Just like there is no greatest natural number (you can always add 1 to any number you think is the greatest), then there is no smallest rational number greater than 0. If we pick one, we can always find a smaller one. Now, let's open up our number line to infinity (negative and positive). By the same reasoning, there is no largest negative rational number going left on our number line and no largest positive rational number going right on our number line.

Next, consider this. Given $\frac{1}{3}$ and $\frac{1}{2}$. How many rational numbers are there between these two rational numbers? We can find one immediately. We take the arithmetic mean of $\frac{1}{3}$ and $\frac{1}{2}$ and since we have already done this, we know the answer is $\frac{5}{12}$, a rational number. Let's continue this process. Between $\frac{1}{3}$ and $\frac{5}{12}$ we can find another rational number by calculating the arithmetic mean of these two numbers:

$$\frac{1}{3}+\frac{5}{12}=\frac{4}{12}+\frac{5}{12}=\frac{9}{12}\div 2=\frac{9}{12}\times\frac{1}{2}=\frac{9}{24}$$

$\frac{9}{24}$ is between $\frac{1}{3}$ and $\frac{5}{12}$. Note also that $\frac{9}{24}$ is between $\frac{1}{3}$ and $\frac{1}{2}$. Between $\frac{1}{3}$ and $\frac{9}{24}$ we can find another rational number by again calculating the arithmetic mean:

$$\frac{1}{3}+\frac{9}{24}=\frac{8}{24}+\frac{9}{24}=\frac{17}{24}\div 2=\frac{17}{24}\times\frac{1}{2}=\frac{17}{48}$$

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$\frac{17}{48}$ is between $\frac{1}{3}$ and $\frac{9}{24}$. Note also that $\frac{17}{48}$ is between $\frac{1}{3}$ and $\frac{1}{2}$. I don't think I need to go any farther with this. You get the idea. *Between any two rational numbers there are an infinite number of rational numbers*. Our number line is *literally packed* with rational numbers. Because of this we can also assert this statement: We cannot calculate a rational number that comes *immediately after* any given rational number. Try it. What is the rational number that comes immediately after $\frac{1}{3}$? Get out a piece of paper and start calculating. Pick a rational number after $\frac{1}{3}$ (say $\frac{5}{12}$) and start calculating the arithmetic mean between $\frac{1}{3}$ and that number. You will find yourself doing the same thing as I illustrated above. If you still think you can find that number, I recommend that you start cutting down all the forests in the world. All that wood still would not supply you with enough paper to do the calculations. And, when you are in your waning years (approaching 100), you still would be calculating but never finding that illusive number. Because of this packing property, mathematicians denote the set of rational numbers as being *everywhere dense*.

Infinity strikes again. The natural numbers are infinite in scope and so are the prime numbers (both sequences tend to infinity). Now we have encountered a new twist on infinity; *the concept of limitless density*. There is no distance so small that within this distance and $\frac{1}{3}$ there are no other rational numbers. Mathematicians express this by saying that $\frac{1}{3}$ is a *condensation point* of the set of rational numbers. Of course, in this context, every rational number is a condensation point.

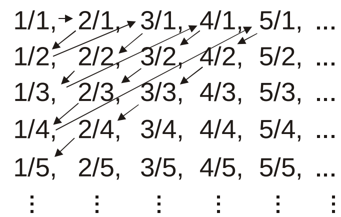
If you think your mind has been stretched in considering infinite packing and condensation points, then buckle your seat beat and shoulder strap. Put on your crash helmet. Call the emergency vehicles. Notify your insurance agent. Write your will. Prepare to meet thy God. Get ready for the ride of your life!

In spite of all that we have said so far it is possible to enumerate all the rational numbers in one sequence. Unlike the set of natural numbers or the set of integers, this sequence is not ordered from smallest to largest. Take a long look at this arrangement:

$$\begin{array}{cccccc} 1/1, & 2/1, & 3/1, & 4/1, & 5/1, & \ldots \\ 1/2, & 2/2, & 3/2, & 4/2, & 5/2, & \ldots \\ 1/3, & 2/3, & 3/3, & 4/3, & 5/3, & \ldots \\ 1/4, & 2/4, & 3/4, & 4/4, & 5/4, & \ldots \\ 1/5, & 2/5, & 3/5, & 4/5, & 5/5, & \ldots \\ 1/6, & 2/6, & 3/6, & 4/6, & 5/6, & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

This enumeration continues to the right *ad infinitum* and it also continues down *ad infinitum*. The pattern should be obvious. Anyone could write as many entries as one wished.

Now we must arrange these rational numbers into a single sequence. We can do this as follows. Start in row 1, column 1. This is our first number. Our second number is in row 1, column 2. Our third number is in row 2, column 1. Our fourth number is in row 1, column 3. Our fifth number is in row 2, column 2. Our sixth number is in row 3, column 1, etc. We are creating a "zig-zag" sequence pictured as follows:



In this way, every rational number in our arrangement will be listed:

$$\frac{1}{1},\quad \frac{2}{1},\frac{1}{2},\quad \frac{3}{1},\frac{2}{2},\frac{1}{3},\quad \frac{4}{1},\frac{3}{2},\frac{2}{3},\frac{1}{4},\quad \frac{5}{1},\frac{4}{2},\frac{3}{3},\frac{2}{4},\frac{1}{5},\ldots$$

Anyone could continue this sequence knowing the rule of its construction. You do not even need to follow the "zig-zags." Note the groupings above. The sum of the numerator and denominator in the first group is 2 and the sum of the numerator and denominator for each term in the second group is 3. For the third group, the sum of the numerator and denominator for each term is 4. For the fourth group, the sum of the numerator and denominator for each term is 5. For the fifth group, the sum of the numerator and denominator for each term is 6. So, for the sixth group, the sum of the numerator and denominator for each term is 7. This group starts with $\frac{6}{1}$ and ends with $\frac{1}{6}$. The numerators decrease by 1 and the denominators increase by 1. The sixth group looks like this:

$$\frac{6}{1},\frac{5}{2},\frac{4}{3},\frac{3}{4},\frac{2}{5},\frac{1}{6}$$

Every succeeding group can be determined in the same fashion. Thus we have established an infinite sequence for the set of rational numbers. Of course, we are going to repeat some numbers (e.g., $\frac{2}{2}$ and $\frac{3}{3}$ reduce to 1). We will add to our rule: *Those rational numbers that can be simplified to lower terms will be left out*. Now our sequence looks like this:

$$\frac{1}{1},\frac{2}{1},\frac{1}{2},\frac{3}{1},\frac{1}{3},\frac{4}{1},\frac{3}{2},\frac{2}{3},\frac{1}{4},\frac{5}{1},\frac{1}{5},\ldots$$

This sequence can also be continued *ad infinitum*. I can state what the first, second, third, fourth, etc. numbers are in the sequence. That means that this sequence can be numbered or counted. Using mathematical terminology, the set of rational numbers can be *enumerated* (i.e., they are *countable*). As a side note, we can include the associated *negative* rational numbers (after every positive rational number) in this list and still be able to count them off one by one.

$$\frac{1}{1},-\frac{1}{1},\frac{2}{1},-\frac{2}{1},\frac{1}{2},-\frac{1}{2},\frac{3}{1},-\frac{3}{1},\frac{1}{3},-\frac{1}{3},\frac{4}{1},-\frac{4}{1},\frac{3}{2},-\frac{3}{2},\frac{2}{3},-\frac{2}{3},\frac{1}{4},-\frac{1}{4},\frac{5}{1},-\frac{5}{1},\frac{1}{5},-\frac{1}{5},\ldots$$

Here is what is astonishing about this state of affairs. In spite of the fact that the rational numbers are everywhere dense, in some sense *there are just as many rational numbers as there are natural numbers*. Do not let this statement slip by your cognition.

We can match every rational number to a counting number as follows:

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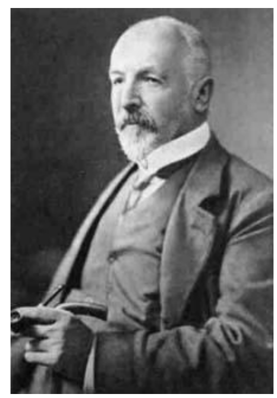
$$\frac{1}{1}, -\frac{1}{1}, \frac{2}{1}, -\frac{2}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{1}, -\frac{3}{1}, \frac{1}{3}, -\frac{1}{3}, \frac{4}{1}, -\frac{4}{1}, \ldots$$

$$\begin{matrix} \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{matrix}$$

The German mathematician Georg Cantor (1845-1918) was the first to develop this idea of matching elements in a set with the natural numbers. This concept is called *one-to-one correspondence*. We use this idea unconsciously with finite sets. For example, consider a room full of boys and girls. Tell the group to pair off (one boy to one girl). Don't let the great New York Yankee catcher Yogi Berra do this, though. He will tell the group to "pair off in threes." By doing this, we are engaging the concept of one-to-one correspondence. *Without counting*, you will immediately know if the number of boys equals the number of girls if one boy or one girl does not have a partner.

Likewise, with the set of rational numbers, we have matched each one with a natural number. Every rational number can be matched with a "date" (i.e., with an associated natural number). If we could count the number of elements in these sets (we cannot since they extend to infinity), then the number of elements in each set would be the same. Cantor denoted sets with the same number of elements as sets with the same *power*. Today, mathematicians tend to replace the word power with the word *cardinality*.[7] But, *we really do not need to count*. We have shown that a one-to-one correspondence exists between the two sets. Cantor denoted this uncountable number as $\aleph_o$ or aleph-null (aleph is the first letter in the Hebrew alphabet and null means "zero magnitude") and he called the set of rational numbers *denumerable*.[8] Cantor called $\aleph_o$ a *transfinite number* because this number "transcends the finite."



George Cantor (Public Domain)

Now ponder this. In this sense (the sense of one-to-one correspondence), the set of natural numbers is as numerous as the set of rational numbers. *This is so in spite of the fact that the counting numbers are a subset (i.e., a part) of the rational numbers.* In other words, the natural numbers are contained in the rational numbers ($\mathbb{N} \subset \mathbb{Z}$). As we can see from inspecting the number line, counting numbers are "indivisible." The rational numbers are "everywhere dense." The rational numbers are packed all over the number line while the counting numbers appear like needles in a haystack.

What do we make of this? We must "tread softly" before infinity. In the finite world, the whole is greater than the part (Euclid's Axiom 5). But, in the world of infinite sets, the whole can be equal to the part. There is an unapproachable chasm between the realm of the finite and the realm of the infinite in mathematics. Yet, God has gifted our minds with the ability to grasp the infinite; albeit this understanding is limited, not exhaustive.

This little truth should remind us of a far greater truth. There is an unapproachable chasm between the infinite God and finite man. Yet, God has breached that chasm in His Son, the God-man Jesus Christ. In Christ, the finite and the infinite meet in gracious mercy. In Christ, we can grasp, in a limited way, the nature of the eternal, infinite, and triune God of majesty and wonder. In Christ, we tread softly before the transcendent God.

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[7] Cardinal comes from a Latin word meaning "principal or pivotal or serving as a hinge." In mathematics, the cardinality concept indicates quantity (e.g., how much? This set has 3 elements). The *ordinal* concept indicates order (e.g., How far along in a sequence? First, second, third, etc.)

[8] Denumerable comes from the Latin and means "to count."