Let It Snow...

by James D. Nickel

One of the most fascinating studies in mathematics is the study of the nature of infinite sequences. A sequence is list of numbers that follow a definite pattern. A sequence can be finite or infinite. For example, the sequence $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\right\}$ is an example of an infinite sequence. What is the pattern? If we let s_n and s_{n+1} represent two successive terms in this sequence, then $s_{n+1} = \frac{1}{2}s_n$. Mathematicians call this sequence a geometric sequence. In general, in a geometric sequence, $s_{n+1} = ks_n$, where k is called the "common ratio." Our familiar set of natural numbers, $\{1, 2, 3, \ldots\}$, is an example of an arithmetic sequence. In general, in an arithmetic sequence, $s_{n+1} = s_n + k$, where k is called the "common difference." The common difference in the sequence of natural numbers is 1.

> Note: This essay is extracted from a Lesson from the forthcoming textbook *Mathematics: Building on Foundations*.

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It is the study of *infinite* geometric sequences that is the most fascinating, especially when you add each term and then consider the sum of all terms, i.e., an infinite sum. In symbols, $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ as $n \to \infty$. The sum of an infinite geometric sequence is called an infinite geometric series. Mathematicians have developed a shorthand notation to express this infinite sum: $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ as $n \to \infty$ $\Leftrightarrow S_n = \lim_{n\to\infty}\sum_{k=1}^{n} a_k$. The symbol "lim" is shorthand hand for "limit" meaning "threshold." Σ is the capital Greek letter sigma and it represents "finding the sum." Rhetorically, $S_n = \lim_{n\to\infty}\sum_{k=1}^{n} a_k$ represents "the sum of an infinite sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ as $n \to \infty$." More specifically, $S_n = \lim_{n\to\infty}\sum_{k=1}^{n} a_k$ means "we calculate the sum of each successive term, a_k, of an infinite sequence."

Let's put this into practice. Given the sequence $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\right\}$, what is its sum? We want to find $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^k} + \ldots$. Note that the general representation for each term is $\frac{1}{2^k}$ where k varies in arithmetic sequence (common difference = 1) from 1 to infinity. Using "sigma notation," we want to find $S_n = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2^k}$. How do we find this sum? Can we add an infinite number of numbers? Technically, no, but we can use our "little grey cells" (a favorite phrase used by Hercule Poirot, the famous fictional Belgian detective created by Agatha Christie) and add an infinite sum "in a snap."

Equation 1. $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^k} + \ldots$

Now, multiply both members of this equation by 2. We get:

Equation 2. $2S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{k-1}} + \ldots$

Now, we perform the mathematical "magic act." We subtract Equation 1 from Equation 2. We get: $S_n = 1$. Why? $2S_n - S_n = S_n$ and $\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{k-1}} + \ldots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^k} + \ldots\right) = 1$. Too easy, isn't it?

In general, we can develop a formula to find the sum of any infinite geometric series. In order to find to this formula, we must first determine the formula for the sum of a *finite* geometric series. We consider the sum, S_n, of first *n* terms of a finite geometric sequence where the first term is a_1 and the common multiplier (or common ratio) is *k*. Symbolically, we have:

Equation 1. $S_n = a_1 + a_1k + a_1k^2 + a_1k^3 + \ldots + a_1k^{n-1}$. Make sure that you see this. Let $a_1 = \frac{1}{2}$ and $k = \frac{1}{2}$ and this equation becomes:

$$S_n = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^3 + \ldots + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} =$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$ (this should look familiar)

Given Equation 1, we multiply both members by k. We get:

Equation 2. $kS_n = a_1k + a_1k^2 + a_1k^3 + \ldots + a_1k^n$

Next, we subtract Equation 1 from Equation 2. We get:

Equation 3. $S_n - kS_n = a_1 - a_1k^n$. Factoring the left member, we get:

Equation 4. $S_n(1-k) = a_1 - a_1k^n$. Dividing both members by k, we get:

Equation 5. $S_n = \frac{a_1 - a_1k^n}{1-k} = \frac{a_1\left(1-k^n\right)}{1-k}$

Hence, $S_n = \frac{a_1\left(1-k^n\right)}{1-k}$ is our formula for finding the *finite* sum of a geometric series. To find the sum, all we need to know is the first term, a_1, and the common ratio, k. Before we continue, are there any conditions that we must place on this formula for it to work? Since we cannot divide by 0, then $k \neq 1$.

What happens when k = 1? Our formula does not work but when we consider $S_n = a_1 + a_1k + a_1k^2 + a_1k^3 + \ldots + a_1k^{n-1}$ and let k = 1, we get:

$$S_n = a_1 + a_1 + a_1 + a_1 + \ldots a_1 \left(\text{n times}\right)$$

This equation represents repeated addition of the same number (i.e., multiplication). Hence, $S_n = na_1$ if k = 1.

Let's find S_5 when $a_1 = \frac{1}{2}$ and $k = \frac{1}{2}$. By the formula, we get:

Let It Snow...

by James D. Nickel

$$S_5 = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^5\right)}{\frac{1}{2}} = 1-\left(\frac{1}{2}\right)^5 = 1-\frac{1}{32} = \frac{31}{32}$$

We now have our formula, $S_n = \frac{a_1\left(1-k^n\right)}{1-k}$, to find the sum of a *finite* geometric sequence. If we let n → ∞, what happens? In symbols, we want to find $S_n = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{a_1\left(1-k^n\right)}{1-k}$. In our formula, the only place where *n* occurs is in the term k^n. Hence, we need to determine what happens to k^n as n → ∞. Or, we want to find $\lim_{n\to\infty} k^n$. We must consider all possible values of k.

As we have seen, when k = 1, $S_n = na_1$. $S_n = \lim_{n\to\infty} na_1$ is a very large number. Since this sum has no limiting value (or threshold), then mathematicians call it *divergent*. When k = 1, our formula will *not* give us a "limiting value."

Let's consider k > 1. As *n* gets larger and larger, k^n gets larger and larger. For example, Let k = 3. We have: 3^1, 3^2, 3^3, … 3^n. Hence, $\lim_{n\to\infty} k^n$ cannot be determined if r ≥ 1. It is divergent.

Now let's consider k ≤ -1. As *n* gets larger and larger, what happens to k^n? For example, let k = -3. We have: (-3,) $(-3)^2$, $-(3)^3$, … $(-3)^n$. If *n* is odd, $(-3)^n$ is a large negative number. If *n* is even, $(-3)^n$ is a large positive number. Hence, $\lim_{n\to\infty} k^n$ cannot be determined, or is divergent, if r ≤ -1.

From this analysis, we are left with a range of numbers for k: -1 < k < 1. If k = 0, then $S_n = \frac{a_1\left(1-k^n\right)}{1-k} = \frac{a_1}{1} = a_1$.

If k = $\frac{1}{2}$, what is $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n$? We note that as n gets larger and larger, $\left(\frac{1}{2}\right)^n$ gets smaller and smaller; it approaches 0 as a threshold. Hence, $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$. Mathematicians say that $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n$ *converges* to 0.

Let's now consider k = $-\frac{1}{2}$. What is $\lim_{n\to\infty}\left(-\frac{1}{2}\right)^n$? Consider this table:

n	$\left(-\frac{1}{2}\right)^n$
1	$-\frac{1}{2}$
2	$\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

Let It Snow…

by James D. Nickel

n	$\left(-\frac{1}{2}\right)^n$
3	$\left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$
4	$\left(-\frac{1}{2}\right)^4 = \frac{1}{16}$
5	$\left(-\frac{1}{2}\right)^5 = -\frac{1}{32}$
⋮	⋮
n	$\left(-\frac{1}{2}\right)^n$

We note that the values of $\left(-\frac{1}{2}\right)^n$ are *converging* on 0 from two directions, negative and positive. Hence, $\lim_{n\to\infty}\left(-\frac{1}{2}\right)^n = 0$.

So, if -1 < k < 1, the k^n term in the formula $S_n = \frac{a_1\left(1-k^n\right)}{1-k}$ *converges* to 0 as *n* gets very, very large. Hence, $S_n = \frac{a_1\left(1-0\right)}{1-k} = \frac{a_1}{1-k}$ as *n* gets very, very large. In symbols:

$$\lim_{n\to\infty} S_n = \frac{a_1\left(1-k^n\right)}{1-k} = \frac{a_1}{1-k} \text{ if } -1 < k < 1$$

This is our formula! How simple it is, given the conditions for k:

$$S_n = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{a_1\left(1-k^n\right)}{1-k} = \frac{a_1}{1-k} \text{ iff } -1 < k < 1.$$

Hence, an infinite geometric series *converges* to the limit $\frac{a_1}{1-k}$ iff -1 < k < 1 where a_1 = the first term of the series and k = common ratio.

Remember $S_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}+\ldots$? Let's use our formula. $a_1 = \frac{1}{2}$ and k = $\frac{1}{2}$. Hence, since k meets the conditions (-1 < k < 1), then $S_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}+\ldots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$. Bingo!

Let It Snow...

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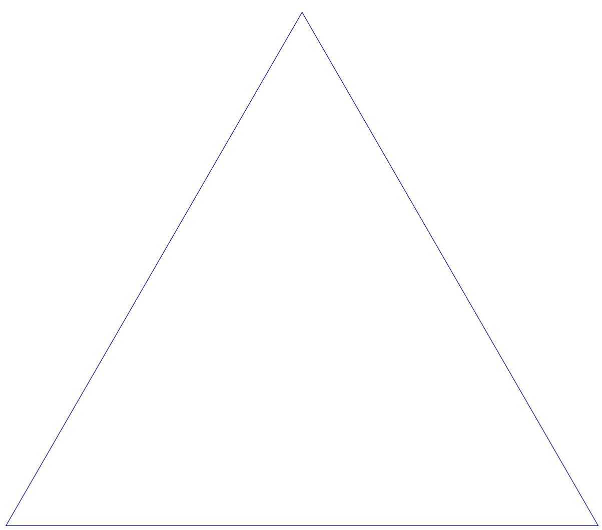
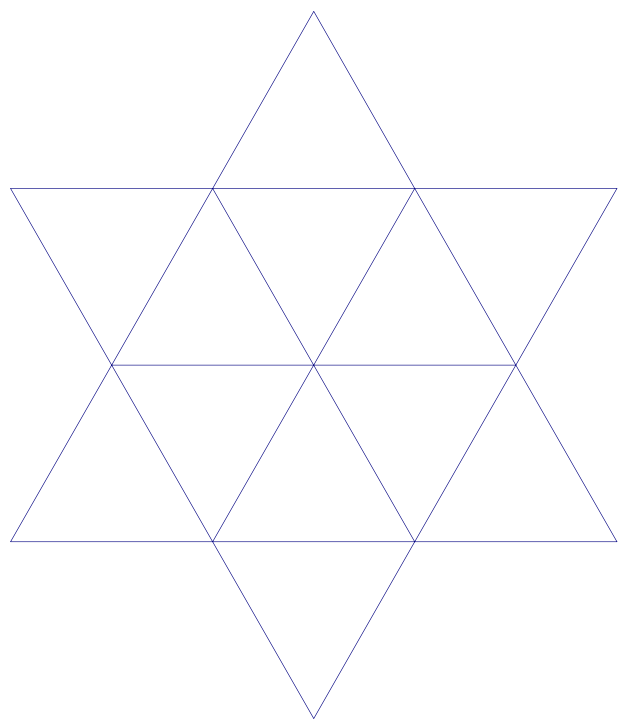
The infinite geometric series "crops up" all over in mathematics. For example, $\frac{1}{3} = 0.\overline{3}$ represents an infinite geometric series. $0.\overline{3} = \frac{3}{10} + \frac{3}{10}\left(\frac{1}{10}\right) + \frac{3}{10}\left(\frac{1}{10}\right)^2 + \frac{3}{10}\left(\frac{1}{10}\right)^3 + \ldots + \frac{3}{10}\left(\frac{1}{10}\right)^{n-1} + \ldots$. Note that $a_1 = \frac{3}{10}$ and $k = \frac{1}{10}$. What is the limiting value, or threshold, of this infinite geometric series? Using our formula, $S_n = \frac{a_1}{1-k} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{3}{9} = \frac{1}{3}$!

In the 20th century, a new branch of geometry was discovered. It is called fractal geometry.[1] In 1975, the French mathematician Benoit Mandelbrot (1924-) used this word to describe this type of geometry. The Koch snowflake (or Koch star) is a mathematical curve and one of the earliest "fractal" curves to have been described. It appeared in a 1904 paper entitled "On a continuous curve without tangents, constructible from elementary geometry" by the Swedish mathematician Helge von Koch (1870-1924). The study of this fascinating curve requires working with a geometric series that is infinite.

To construct the Koch snowflake, we start with an equilateral triangle. To make the calculations easy, we let the area of this triangle be one square unit.



Equilateral triangle, Area = 1 unit²



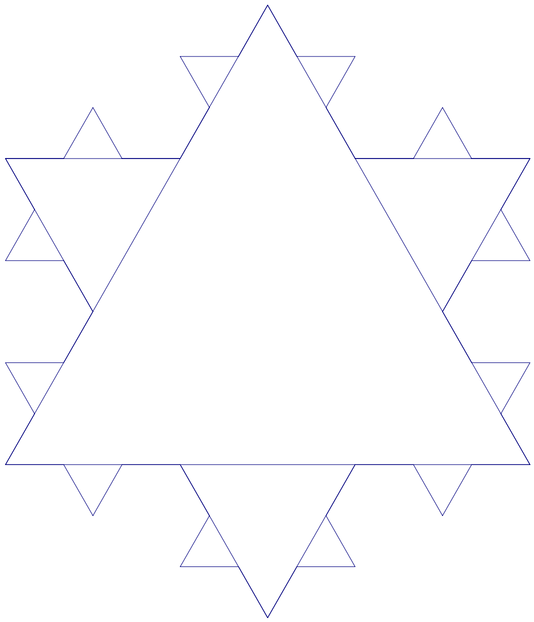
Koch snowflake, iteration 1

Each succeeding figure (or iteration figure) is made by dividing each side of the original equilateral triangle into three equal parts and then adding a triangular piece on each of the center pieces of the side. After the first iteration, the total area of the Koch snowflake is now: $1 + \frac{3}{9}$ (three equilateral tri-

[1] Fractal comes from fraction and means "to break."

angles have been added each with the area of $\frac{1}{9}$).

For our second iteration, the area of the Koch snowflake now becomes $1+\frac{3}{9}+\frac{12}{81}$. We added 12 equilateral triangles each with the area of $\frac{1}{81}$.

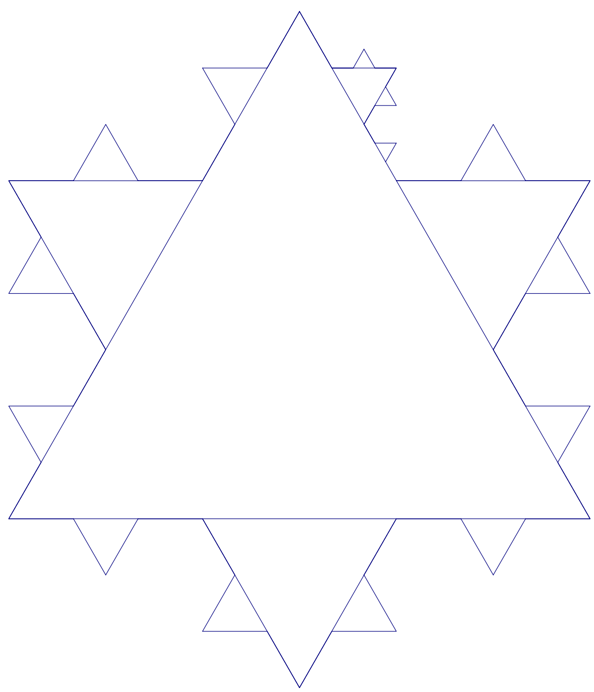


Koch snowflake, iteration 2

For our third iteration, the area of the Koch snowflake now becomes $1+\frac{3}{9}+\frac{12}{81}+\frac{48}{729}$. We added 48 equilateral triangles each with the area of $\frac{1}{729}$. If we continue these iterations, the area of the Koch snowflake would be: $1+\frac{3}{9}+\frac{12}{81}+\frac{48}{729}+\cdots$ Do you see the pattern? What is the area of the Koch snowflake or this infinite sum?

Here is how we use our "little grey cells" to write this sum in another form so that we can uncover a hidden infinite series so that we can determine a_1, the first term, and k, the common ratio:

$$1+\frac{3}{9}+\frac{12}{81}+\frac{48}{729}+\cdots=$$



Koch snowflake, start of iteration 3

Let It Snow…

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$$1+3\left(\frac{4^0}{9^1}\right)+3\left(\frac{4^1}{9^2}\right)+3\left(\frac{4^2}{9^3}\right)+\cdots=$$

$$1+3\left(\frac{4^0}{9^1}+\frac{4^1}{9^2}+\frac{4^2}{9^3}+\cdots\right)=$$

$$1+\frac{3}{4}\left(\frac{4^1}{9^1}+\frac{4^2}{9^2}+\frac{4^3}{9^3}+\cdots\right)=$$

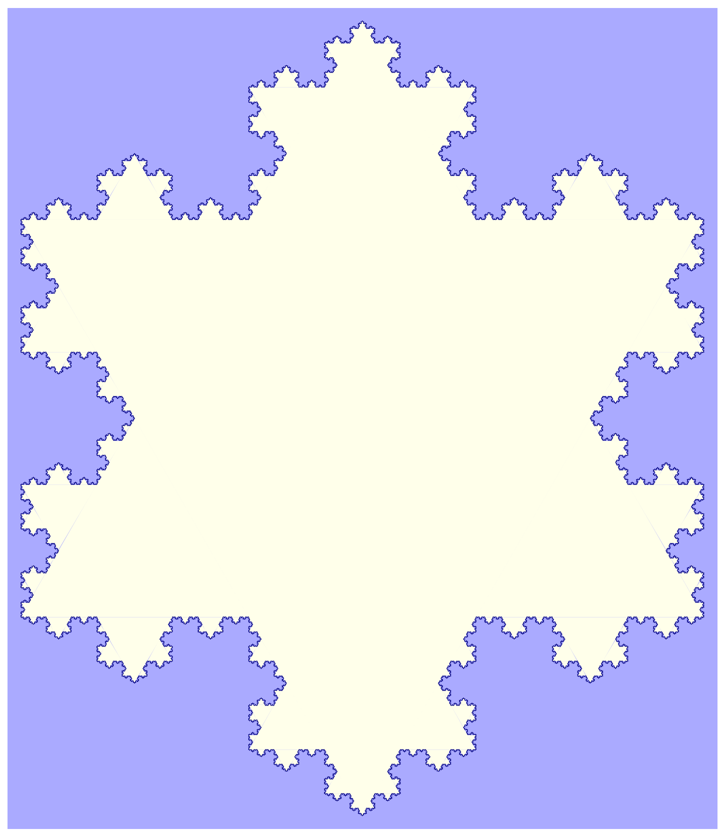
$$1+\frac{3}{4}\left(\frac{4^1}{9^1}+\frac{4^2}{9^2}+\frac{4^3}{9^3}+\cdots\right)=$$

Within the parentheses, we have $\frac{4^1}{9^1}+\frac{4^2}{9^2}+\frac{4^3}{9^3}+\cdots$ where $a_1 = \frac{4}{9}$ and $k = \frac{4}{9}$. Hence, since $-1 < k < 1$, then $\frac{4^1}{9^1}+\frac{4^2}{9^2}+\frac{4^3}{9^3}+\cdots=\frac{\frac{4}{9}}{1-\frac{4}{9}}=\frac{\frac{4}{9}}{\frac{5}{9}}=\frac{4}{5}$. Therefore, $1+\frac{3}{4}\left(\frac{4^1}{9^1}+\frac{4^2}{9^2}+\frac{4^3}{9^3}+\cdots\right)=1+\left(\frac{3}{\cancel{4}}\right)\left(\frac{\cancel{4}}{5}\right)=1\frac{3}{5}=1.6$.

Recall that we started with an equilateral triangle whose area was 1 square unit. Applying the "Koch iteration" to infinity generates the Koch snowflake with a *finite* area of 1.6 square units. It should not surprise us that, since $\frac{1}{3}=0.\overline{3}$, the sum of an *infinite* series of numbers is a *finite* number or, to put it mathematically, an infinite series converges to a limiting value.

Fractal geometry is about these iterative procedures and the properties obtained thereby. The marvel of the Koch snowflake, pictured at right, is that you can zoom to any region, and what you see is a part of the same figure. Keep zooming, *ad infinitum,* and the same figure appears again and again and again …

The Koch snowflake is one example of fractal geometry, the geometry of the *infinite iteration* of self-similar figures.



The beautiful Koch Snowflake

Before we conclude our tour of snow land, let's ask, "What is the perimeter of the Koch snowflake?

We let the perimeter of the original equilateral triangle be 1 to make our calculations easier. This means that the length of each side is $\frac{1}{3}$.

Let It Snow...

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After the first iteration, the perimeter is: $1+\frac{1}{3}$. Inspecting the figure, we took away $\frac{1}{9}$ (3 sides of length $\frac{1}{3}$) and added back $\frac{6}{9}=\frac{2}{3}$. Our net gain in perimeter is therefore $\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$. Make sure that you understand this.

After the second iteration, the perimeter is: $1+\frac{1}{3}+\frac{12}{27}$. Although I will not explain this, you should be able to see that the net gain in perimeter with the second iteration is $\frac{12}{27}$. After the third iteration, the perimeter is: $1+\frac{1}{3}+\frac{12}{27}+\frac{48}{81}$. Again, we have the beginning of an infinite series although it is a different from the one we discovered when we studied the area of the Koch snowflake. What is the pattern? To find it, follow along:

$$1+\frac{1}{3}+\frac{12}{27}+\frac{48}{81}+\ldots=$$

$$1+\frac{1}{3}+\frac{4\cdot 3}{3^3}+\frac{16\cdot 3}{3^4}+\ldots=$$

$$1+\frac{2^0}{3^1}+\frac{2^2}{3^2}+\frac{2^4}{3^3}+\ldots=$$

$$1+\frac{4^0}{3^1}+\frac{4^1}{3^2}+\frac{4^2}{3^3}+\ldots=$$

$$1+\frac{1}{3}\left(\frac{4^0}{3^0}+\frac{4^1}{3^1}+\frac{4^2}{3^2}+\ldots\right)=$$

$$1+\frac{1}{3}\left[\left(\frac{4}{3}\right)^0+\left(\frac{4}{3}\right)^1+\left(\frac{4}{3}\right)^2+\ldots\right]$$

Within the brackets, we have an infinite geometric series $\left(\frac{4}{3}\right)^0+\left(\frac{4}{3}\right)^1+\left(\frac{4}{3}\right)^2+\ldots$ where a = 1 and k $=\frac{4}{3}$. Since k ≥ 1, then the series in brackets is *divergent*. Hence, the perimeter of the Koch snowflake, as the iterations approach infinity, gets larger and larger and larger …

With the Koch snowflake, its area converges to 1.6 square units, but its perimeter diverges and increases indefinitely as the iterations go to infinity. *This is a dazzling conclusion.* Take time to ponder it carefully.