CALCULATING SQUARE ROOTS BY HAND

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Before the invention of electronic calculators, students followed two algorithms to approximate the square root of any given number. First, we are going to investigate the ancient Babylonian algorithm (or method) for calculating the square root of any positive integer.[1] Second, we will investigate another algorithm that was commonly used to calculate square roots in textbooks before the invention of the electronic calculator. There is a lot of pertinent mathematics revealed in these methods and it is unfortunate that most "calculator-savvy" students are unaware of them.[2]

THE BABYLONIAN ALGORITHM

The Babylonian method to calculate square roots is a recursive algorithm. A *recursive algorithm* is a rule or procedure that can be *repeatedly applied*. Here are the three steps:

Step 1. Estimate the square root of the given positive integer. We will shortly learn how to make good estimates, not wild estimates.

Step 2 (The crux of the method). Calculate the average of that guess and the given positive integer divided by that guess.

Step 3. Use your answer to Step 2 as your new guess and repeat Step 2 (this is the "recursion") until the desired degree of accuracy is obtained.

Recall that *average* (or *mean*) is a measure of the central tendency of a group of numbers. Since we are taking the average of two numbers, the "central tendency" of these two numbers will be the number halfway between. To find the average of any two numbers, 3 and 5 for example, we add them and then divide the sum by 2; i.e., $\frac{3+5}{2}=\frac{8}{2}=4$.[3] Hence, the average of 3 and 5 is 4. This makes perfect sense because, on a numberline, 4 is halfway between 3 and 5.

Before we apply this method, let's see if we can refine our skills of guessing. Suppose we want to find the square root of a positive integer. We know that the lengths of the digits in a positive integer can be either odd or even. We are going to make use of this fact to help us make accurate first guesses of the square root of any positive integer.

It will help us to determine a pattern if we look at this table.

Positive Integer	*Number of digits*	*Square root*	*Number of digits before the decimal point*
1	1	1	1
4	1	2	1

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[1] Modern calculators can find the square root of numbers with a single push of a button. *What is programmed into these calculators to find the square root is this Babylonian algorithm!*

[2] Electronic calculators (with their "square root" key) are now so popular that teaching square root algorithms is almost *passé* (a French word meaning "no longer fashionable"). In spite of this, understanding why the traditional algorithm works and watching the Babylonian algorithm quickly find the square root (why it works is proven in Calculus, as we shall shortly note) is beneficial for refining one's understanding of mathematical processes (particularly, number sense).

[3] If, in a fraction, we are adding numbers in the numerator, we want to do that *first* because *the fraction bar acts as a parenthesis.* $\frac{3+5}{2}=(3+5)\div 2$. Note carefully that $\frac{3+5}{2}\neq 3\div 2+5$.

Positive Integer	*Number of digits*	*Square root*	*Number of digits before the decimal point*
8	1	≈ 2.83	1
10	2	≈ 3.16	1
40	2	≈ 6.32	1
80	2	≈ 8.94	1
100	3	10	2
400	3	20	2
800	3	≈ 28.3	2
1000	4	≈ 31.6	2
4000	4	≈ 63.2	2
8000	4	≈ 89.4	2
10,000	5	100	3
40,000	5	200	3
80,000	5	≈ 283	3
100,000	6	≈ 316	3
400,000	6	≈ 632	3
800,000	6	≈ 894	3
1,000,000	7	1000	4
4,000,000	7	2000	4
8,000,000	7	≈2830	4
10,000,000	8	≈3160	4
40,000,000	8	≈6320	4
80,000,000	8	≈ 8940	4
100,000,000	9	10,000	5
400,000,000	9	20,000	5
800,000,000	9	≈ 28,300	5
1,000,000,000	10	≈31,600	5

Let's now inspect the following table of squares and square roots:

Number	*Square*	*Square Root*	***Number***	*Square*	*Square Root*	***Number***	*Square*	*Square Root*
1	1	1	**34**	1156	5.831	**67**	4489	8.185
2	4	1.414	**35**	1225	5.916	**68**	4624	8.246
3	9	1.732	**36**	1296	6	**69**	4761	8.307
4	16	2	**37**	1369	6.083	**70**	4900	8.367
5	25	2.236	**38**	1444	6.164	**71**	5041	8.426
6	36	2.449	**39**	1521	6.245	**72**	5184	8.485
7	49	2.646	**40**	1600	6.325	**73**	5329	8.544
8	64	2.828	**41**	1681	6.403	**74**	5476	8.602
9	81	3	**42**	1764	6.481	**75**	5625	8.660
10	100	3.162	**43**	1849	6.557	**76**	5776	8.718

Number	*Square*	*Square Root*	***Number***	*Square*	*Square Root*	***Number***	*Square*	*Square Root*
11	121	3.317	**44**	1936	6.633	**77**	5929	8.775
12	144	3.464	**45**	2025	6.708	**78**	6084	8.832
13	169	3.606	**46**	2116	6.782	**79**	6241	8.888
14	196	3.742	**47**	2209	6.856	**80**	6400	8.944
15	225	3.873	**48**	2304	6.928	**81**	6561	9
16	256	4	**49**	2401	7	**82**	6724	9.055
17	289	4.123	**50**	2500	7.071	**83**	6889	9.110
18	324	4.243	**51**	2601	7.141	**84**	7056	9.165
19	361	4.359	**52**	2704	7.211	**85**	7225	9.220
20	400	4.472	**53**	2809	7.280	**86**	7396	9.274
21	441	4.583	**54**	2916	7.348	**87**	7569	9.327
22	484	4.690	**55**	3025	7.416	**88**	7744	9.381
23	529	4.796	**56**	3136	7.483	**89**	7921	9.434
24	576	4.899	**57**	3249	7.550	**90**	8100	9.487
25	625	5	**58**	3364	7.616	**91**	8281	9.539
26	676	5.099	**59**	3481	7.681	**92**	8464	9.592
27	729	5.196	**60**	3600	7.746	**93**	8649	9.644
28	784	5.292	**61**	3721	7.810	**94**	8836	9.695
29	841	5.385	**62**	3844	7.874	**95**	9025	9.747
30	900	5.477	**63**	3969	7.937	**96**	9216	9.798
31	961	5.568	**64**	4096	8	**97**	9409	9.849
32	1024	5.657	**65**	4225	8.062	**98**	9604	9.899
33	1089	5.745	**66**	4356	8.124	**99**	9801	9.950
						100	10,000	10

Look especially at the perfect squares (the numbers in **red** and the square columns).[4] Do you see a relationship between the number of digits in the perfect square and the number of digits in the square root of the perfect square? Here is the general pattern:

Square roots of perfect squares principle: If any perfect square can be separated into periods of two digits each, beginning with the ones place (from right to left), the number of periods will be equal to the number of digits in the square root of that number. Note also that if the number of digits is *odd*, the left-most period will contain only one digit.

Let's now take careful note of this pattern:

Positive Integer	*Square root*		*Scientific Notation*
$10^0 = 1$	1	=	1×10^0
$10^1 = 10$	≈ 3.16	=	3.16×10^0
$10^2 = 100$	10	=	1×10^1
$10^3 = 1000$	≈ 31.6	=	3.16×10^1
$10^4 = 10{,}000$	100	=	1×10^2
$10^5 = 100{,}000$	≈ 316	=	3.16×10^2

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[4] A perfect square is a positive integer that has an *exact* square root; it is a rational number (more specifically, also a positive integer). Square roots of numbers that are not perfect squares are irrational numbers and therefore we must estimate these roots to a certain precision (or, number of decimal places).

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Positive Integer	*Square root*		*Scientific Notation*
10^6 =1,000,000	1000	=	1×10^3
10^7 =10,000,000	≈ 3160	=	3.16×10^3
10^8 =100,000,000	10,000	=	1×10^4
10^9 =1,000,000,000	≈ 31,600	=	3.16×10^4

Two conclusions follow from these observations:

1. Powers of 10 containing an *odd* number of digits or with *even* exponents are perfect squares.
2. Powers of 10 containing an *even* number of digits or with *odd* exponents are *not* perfect squares.

Since we know $\sqrt{10^0} = \sqrt{1} = 1$ and $\sqrt{10^1} = \sqrt{10} \approx 3.16$, we can immediately determine the square root of *any* power of 10 whose exponent is greater than 1 using these rules:

1. If the exponent is *even*, divide it by 2 and multiply 1 by 10 raised to that new exponent. For example, $\sqrt{10^8} = 1 \times 10^4 = 10,000$.
2. If the exponent is *odd,* subtract 1 from it and then divide the difference by 2. Next, multiply 3.16 by 10 raised to that new exponent. For example, $\sqrt{10^7} \approx 3.16 \times 10^3 = 3160$. Note also that $\frac{7-1}{2} = \frac{6}{2} = 3$.

The same principle can be applied to other numbers. For example, we know that $\sqrt{8} \approx 2.83$ and $\sqrt{80} \approx 8.94$. From these two starting values, we can calculate the square roots of *any* multiple of 8 that is a power of 10. Note the table:

Positive Integer	*Square root*
8	≈ 2.83
80 = 8 × 10^1	≈ 8.94
800 = 8 × 10^2	≈ 28.3
8000 = 8 × 10^3	≈ 89.4
80,000 = 8 × 10^4	≈ 283
800,000 = 8 × 10^5	≈ 894
8,000,000 = 8 × 10^6	≈ 2830
80,000,000 = 8 × 10^7	≈ 8940
800,000,000 = 8 × 10^8	≈ 28,300

For two digit numbers, we know that the square roots will always be 1 digit numbers. The perfect squares, i.e., $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, $25 = 5^2$, $36 = 6^2$, $49 = 7^2$, $64 = 8^2$, and $81 = 9^2$, will help us with our estimations.

For three and four digit numbers, we know that the square roots will always be two digit numbers. To estimate, round the odd digit number to one significant digit and the even digit number to two significant digits. For example, $\sqrt{327} \approx \sqrt{300}$ and $\sqrt{4796} \approx \sqrt{4800}$. Next, estimate the square root of the significant digits (i.e., remove the zeroes). For example, $\sqrt{3} \approx 2$ and $\sqrt{48} \approx 7$. Try to figure these estimates as close as you can. Add one zero for every two zeroes you removed. Why? When you square a power of 10, you double the number of zeroes (e.g., $100^2 = (10^2)^2 = 10^4 = 10,000$). Hence, the estimate of $\sqrt{327} \approx \sqrt{300} \approx 20$ and the estimate of $\sqrt{4796} \approx \sqrt{4800} \approx 70$. *These are your initial guesses that go into the Babylonian algorithm.*

For five and six digit numbers, we know that the square roots will always be three digit numbers. Again, to estimate, round the odd digit number to one significant digit and the even digit number to two significant digits.

Next, estimate the square root of the significant digits (i.e., remove the zeroes). Add one zero for every two zeroes you removed. For example, $\sqrt{78{,}236} \approx \sqrt{80{,}000} \approx 300$ and $\sqrt{485{,}818} \approx \sqrt{490{,}000} \approx 700$.

For seven and eight digit numbers, we know that the square roots will always be four digit numbers. Apply the same procedure as above. For example, $\sqrt{5{,}309{,}878} \approx \sqrt{5{,}000{,}000} \approx 2000$ and $\sqrt{22{,}809{,}019} \approx \sqrt{23{,}000{,}000} \approx$ 5000.

For nine and ten digit numbers, we know that the square roots will always be five digit numbers. Apply the same procedure as above. For example, $\sqrt{682{,}415{,}216} \approx \sqrt{700{,}000{,}000} \approx 20{,}000$ and $\sqrt{3{,}918{,}666{,}154} \approx$ $\sqrt{3{,}900{,}000{,}000} \approx 60{,}000$.

Very rarely will we be called upon the calculate square roots of large numbers that are more than four or five digits in length, but it is always good to give an estimate with the correct number of digits!

Let's only consider the calculation of square roots between 1 and 100. For our first try, let's calculate $\sqrt{5}$. A good first estimate is 2. Applying the Babylonian algorithm (calculate the average of 2 and the 5 divided by 2), we get:

$$\frac{2+\frac{5}{2}}{2} =$$

$$\frac{\frac{4}{2}+\frac{5}{2}}{2} =$$

$$\frac{\frac{9}{2}}{2} = \frac{9}{2} \times \frac{1}{2} = \frac{9}{4}$$

Our first estimate, as a fraction, is $\frac{9}{4}$ or 2.25. Note also that as we found this estimate we calculated the average of $\frac{4}{2}+\frac{5}{2}$, *two numbers that are close to each other.*

Let's apply our algorithm again (the recursive aspect). We start this time with $\frac{9}{4}$ (we will stay with fractional representations so that we can exercise our mastery of fractions). Calculating the average of $\frac{9}{4}$ and the 5 divided by $\frac{9}{4}$, we get:

$$\frac{\frac{9}{4}+\frac{5}{\frac{9}{4}}}{2} =$$

$$\frac{\frac{9}{4}+5\times\frac{4}{9}}{2} =$$

$$\frac{\frac{9}{4}+\frac{20}{9}}{2}=$$

$$\frac{\frac{81}{36}+\frac{80}{36}}{2}=\frac{\frac{161}{36}}{2}=$$

$$\frac{161}{36}\times\frac{1}{2}=\frac{161}{72}$$

Our next estimate is $\frac{161}{72}$ or 2.236 (to four significant figures). Note again carefully that as we found this estimate we calculated the average of $\frac{81}{36}+\frac{80}{36}$, *two numbers very, very close to each other.*

Let's try the algorithm one more time. This will test your mastery of fractions and multiplication *to the limit.* You can do it! Our input this time is $\frac{161}{72}$. Calculating the average of $\frac{161}{72}$ and the 5 divided by $\frac{161}{72}$, we get:

$$\frac{\frac{161}{72}+\frac{5}{\frac{161}{72}}}{2}=$$

$$\frac{\frac{161}{72}+5\times\frac{72}{161}}{2}=$$

$$\frac{\frac{161}{72}+\frac{360}{161}}{2}=$$

$$\frac{\frac{25,921}{11,592}+\frac{25,920}{11,592}}{2}=\frac{\frac{51,841}{11,592}}{2}=$$

$$\frac{51,841}{11,592}\times\frac{1}{2}=\frac{51,841}{23,184}$$

Our next estimate is $\frac{51,841}{23,184}$ or 2.236 (to four significant figures). Four significant figures are sufficient for a very good estimate. *Notice that we obtained this accuracy after only two iterations of the algorithm!* And, again, that as we found this estimate we calculated the average of $\frac{25,921}{11,592}+\frac{25,920}{11,592}$, *two numbers extremely close to each other.*

Starting with a good guess, the Babylonian algorithm will pinpoint the square root of any number with amazing speed and accuracy (usually only two or three iterations are necessary). Because this algorithm is recursive, it is very easy to program it into calculators, spreadsheets, and computers.

How did the ancient Babylonians determine this algorithm? The algorithm is dealing with taking averages of two guesses and that is probably how they determined it. We have to wait until the 17th century when Sir Isaac Newton (1642-1727), a man gifted of God with incredible determination, persistence, and genius, to show (1) why this algorithm works and (2) why it works so fast and so accurately. You will have to wait until you take calculus to discover his answer!

As we have noted, the Babylonian algorithm for finding square roots can easily be programmed into a computer. If you have access to a computer that is running Microsoft Excel (or any spreadsheet program), you see how quickly this algorithm approximates the square root of any number. Spreadsheet programs work on the principle of coordinated "cells."

	A	B	C	D
1				
2				
3				
4				

Like a map, each cell can be located with a fixed "address." For example, A1 is the location of the cell in the first row, first column. Copy the four by four cell above on a piece of paper and answer the following:

1. Write the number 4 in cell A1.
2. Write 15 in cell B4.
3. Write 22 in cell D4.
4. Write the letter "H" in cell C1.
5. Write the letter "E" in cell C2.
6. Write the letter "L" in cell C3.
7. Write the letter "P" in cell C4.

If you have spreadsheet software (this example will follow Microsoft Excel), do the following (or, ask your parent or teacher to assist you):

1. We want to calculate $\sqrt{3}$. Open a new spreadsheet and write the word "Input" in cell A1, "Output" in cell B1 (these two cells serve as "headers").
2. With your mouse, highlight cells A2 to B8. On the menu, click Format/Cells. Under "Category," choose "Number" and select 15 for the number of decimal places. Do not click on the "Use 1000 Separator" box. Click "OK" when you are finished.
3. Write 1 in cell A2 (our first estimate).
4. Write the Babylonian algorithm in cell B2 as follows: =(A2+3/A2)/2. This means that cell B2 equals the results of the algorithm. First, we add the guess (the contents of cell A2) and 3 divided by that guess (the contents of cell A2). Second, you divide that sum by 2; i.e., you are computing the average of the guess and 3 divided by the guess. You should get B2 = 2 (without the trailing zeroes).
5. Write =B2 in cell A3 (our output from the first calculation is now input for the second calculation).
6. Click on cell B2 and copy it (by clicking Ctrl-C). Move your cursor to cell B3 and paste (by clicking Ctrl-V). You should get B3 = 1.75 (without the trailing zeroes).
7. Click on cell A3 and copy it. Move your cursor to cell A4 and paste. Copy and paste cell B3 to B4. You should get B4 = 1.732142857142860
8. Apply the same procedure to cells A7 and B7. You results should look like this:

Input	Output
1.000000000000000	2.000000000000000
2.000000000000000	1.750000000000000
1.750000000000000	1.732142857142860
1.732142857142860	1.732050810014730
1.732050810014730	1.732050807568880
1.732050807568880	1.732050807568880

Note that the output for cells B6 and B7 are identical. This means that the computer has calculated $\sqrt{3}$ accurately to 15 decimal places; i.e., $\sqrt{3}$ = 1.732050807568880

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Wow!

Now, do these problems with your spreadsheet.

1. Using the same methods, calculate $\sqrt{5}$ to 15 decimal places starting with an estimate of 2.
2. Using the same methods, calculate $\sqrt{5}$ to 15 decimal places starting with an estimate of 1.
3. Using the same methods, calculate $\sqrt{27}$ to 15 decimal places starting with an estimate of 5.
4. Using the same methods, calculate $\sqrt{1849}$ to 15 decimal places starting with an estimate of 40. What do you notice?
5. Using the same methods, calculate $\sqrt{815,000}$ to 15 decimal places starting with an estimate of 900.

Use the Babylonian algorithm to find the following square roots to two decimal places (hundredths position). Make an estimate first by asking yourself, "Between what two perfect squares is the number under consideration?" Then, start your algorithm with a positive integer guess.

1. $\sqrt{2}$ Hint: 2 is between 1 and 4. Hence, $\sqrt{2}$ is between 1 and 2.
2. $\sqrt{3}$ Hint: 3 is between 1 and 4. Hence, $\sqrt{3}$ is between 1 and 2.
3. $\sqrt{6}$
4. $\sqrt{7}$
5. $\sqrt{8}$
6. $\sqrt{10}$

Estimate the following square roots. Use the tables in this essay to help.

7. $\sqrt{1420}$
8. $\sqrt{9280}$
9. $\sqrt{22,500}$
10. $\sqrt{30,700}$
11. $\sqrt{2750}$
12. $\sqrt{590}$
13. $\sqrt{51}$
14. $\sqrt{3,250,000}$
15. $\sqrt{900,000}$

Answers are on the next page.

Answers.

1. $\sqrt{2}$ Hint: 2 is between 1 and 4. Hence, $\sqrt{2}$ is between 1 and 2.
   You must start with a guess. Hence, answers will vary based upon the initial guess.
   Let the initial guess be 1.
   Algorithm, first try:

$$\frac{1+\frac{2}{1}}{2}=\frac{1+2}{2}=\frac{3}{2}=1.5$$

   Algorithm, second try:

$$\frac{\frac{3}{2}+\frac{2}{\frac{3}{2}}}{2}=\frac{\frac{3}{2}+2\times\frac{2}{3}}{2}=\frac{\frac{3}{2}+\frac{4}{3}}{2}=\frac{\frac{9}{6}+\frac{8}{6}}{2}=\frac{\frac{17}{6}}{2}=\frac{17}{6}\times\frac{1}{2}=\frac{17}{12}$$

$$\frac{17}{12}\approx 1.417$$

   Algorithm, third try:

$$\frac{\frac{17}{12}+\frac{2}{\frac{17}{12}}}{2}=\frac{\frac{17}{12}+2\times\frac{12}{17}}{2}=\frac{\frac{17}{12}+\frac{24}{17}}{2}=\frac{\frac{289}{204}+\frac{288}{204}}{2}=\frac{\frac{577}{204}}{2}=\frac{577}{204}\times\frac{1}{2}=\frac{577}{408}$$

$$\frac{577}{408}\approx 1.414$$

   We can stop. A good estimate, rounded to the hundredths position, is 1.41.

2. $\sqrt{3}$ Hint: 3 is between 1 and 4. Hence, $\sqrt{3}$ is between 1 and 2.
   Let the initial guess be 1.
   Algorithm, first try:

$$\frac{1+\frac{3}{1}}{2}=\frac{1+3}{2}=\frac{4}{2}=2$$

   Algorithm, second try:

$$\frac{2+\frac{3}{2}}{2}=\frac{\frac{7}{2}}{2}=\frac{7}{2}\times\frac{1}{2}=\frac{7}{4}$$

$$\frac{7}{4}=1.75$$

$$\frac{\frac{7}{4}+\frac{3}{\frac{7}{4}}}{2}=\frac{\frac{7}{4}+3\times\frac{4}{7}}{2}=\frac{\frac{7}{4}+\frac{12}{7}}{2}=\frac{\frac{49}{28}+\frac{48}{28}}{2}=\frac{\frac{97}{28}}{2}=\frac{97}{28}\times\frac{1}{2}=\frac{97}{56}$$

$$\frac{97}{56}\approx 1.732$$

   We can stop. A good estimate, rounded to the hundredths position, is 1.73.

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3. $\sqrt{6}$

   Let the initial guess be 2.

   Algorithm, first try:

   $$\frac{2+\frac{6}{2}}{2}=\frac{2+3}{2}=\frac{5}{2}$$

   $$\frac{5}{2}=2.5$$

   Algorithm, second try:

   $$\frac{\frac{5}{2}+\frac{6}{\frac{5}{2}}}{2}=\frac{\frac{5}{2}+6\times\frac{2}{5}}{2}=\frac{\frac{5}{2}+\frac{12}{5}}{2}=\frac{\frac{25}{10}+\frac{24}{10}}{2}=\frac{\frac{49}{10}}{2}=\frac{49}{10}\times\frac{1}{2}=\frac{49}{20}$$

   $$\frac{49}{20}=2.45$$

   Algorithm, third try:

   $$\frac{\frac{49}{20}+\frac{6}{\frac{49}{20}}}{2}=\frac{\frac{49}{20}+6\times\frac{20}{49}}{2}=\frac{\frac{49}{20}+\frac{120}{49}}{2}=\frac{\frac{2401}{980}+\frac{2400}{980}}{2}=\frac{\frac{4801}{980}}{2}=\frac{4801}{980}\times\frac{1}{2}=\frac{4801}{1960}$$

   $$\frac{4801}{1960}\approx 2.449$$

   We can stop. A good estimate, rounded to the hundredths position, is 2.45.

4. $\sqrt{7}$

   Let the initial guess be 3.

   Algorithm, first try:

   $$\frac{3+\frac{7}{3}}{2}=\frac{\frac{16}{3}}{2}=\frac{16}{3}\times\frac{1}{2}=\frac{16}{6}=\frac{8}{3}$$

   $$\frac{8}{3}=2.\overline{6}$$

   Algorithm, second try:

   $$\frac{\frac{8}{3}+\frac{7}{\frac{8}{3}}}{2}=\frac{\frac{8}{3}+7\times\frac{3}{8}}{2}=\frac{\frac{8}{3}+\frac{21}{8}}{2}=\frac{\frac{64}{24}+\frac{63}{24}}{2}=\frac{\frac{127}{24}}{2}=\frac{127}{24}\times\frac{1}{2}=\frac{127}{48}$$

   $$\frac{127}{48}\approx 2.645$$

   Algorithm, third try:

   $$\frac{\frac{127}{48}+\frac{7}{\frac{127}{48}}}{2}=\frac{\frac{127}{48}+7\times\frac{48}{127}}{2}=\frac{\frac{127}{48}+\frac{336}{127}}{2}=\frac{\frac{16{,}129}{6096}+\frac{16{,}128}{6096}}{2}=\frac{\frac{32{,}257}{6096}}{2}=\frac{32{,}257}{6096}\times\frac{1}{2}=\frac{32{,}257}{12{,}192}$$

$$\frac{32,257}{12,192} \approx 2.645$$

We can stop. A good estimate, rounded to the hundredths position, is 2.65.

5. $\sqrt{8}$

Let the initial guess be 3.

Algorithm, first try:

$$\frac{3+\frac{8}{3}}{2} = \frac{\frac{17}{3}}{2} = \frac{17}{3} \times \frac{1}{2} = \frac{17}{6}$$

$$\frac{17}{6} = 2.8\overline{3}$$

Algorithm, second try:

$$\frac{\frac{17}{6}+\frac{8}{\frac{17}{6}}}{2} = \frac{\frac{17}{6}+8\times\frac{6}{17}}{2} = \frac{\frac{17}{6}+\frac{48}{17}}{2} = \frac{\frac{289}{102}+\frac{288}{102}}{2} = \frac{\frac{577}{102}}{2} = \frac{577}{102} \times \frac{1}{2} = \frac{577}{204}$$

$$\frac{577}{204} \approx 2.828$$

We can stop. A good estimate, rounded to the hundredths position, is 2.83.

6. $\sqrt{10}$

Let the initial guess be 3.

Algorithm, first try:

$$\frac{3+\frac{10}{3}}{2} = \frac{\frac{19}{3}}{2} = \frac{19}{3} \times \frac{1}{2} = \frac{19}{6}$$

$$\frac{19}{6} = 3.1\overline{6}$$

Algorithm, second try:

$$\frac{\frac{19}{6}+\frac{10}{\frac{19}{6}}}{2} = \frac{\frac{19}{6}+10\times\frac{6}{19}}{2} = \frac{\frac{19}{6}+\frac{60}{19}}{2} = \frac{\frac{361}{114}+\frac{360}{114}}{2} = \frac{\frac{721}{114}}{2} = \frac{721}{114} \times \frac{1}{2} = \frac{721}{228}$$

$$\frac{721}{228} \approx 3.162$$

We can stop. A good estimate, rounded to the hundredths position, is 3.16.

Estimate the following square roots. Use the tables in the lesson to help.

7. $\sqrt{1420} \approx \sqrt{1400}$

The square root has two digits.

$\sqrt{14}$ is between 3 and 4 but closer to 4.

So, a good starting estimate would be 37 or 38 (answers may vary).

8. $\sqrt{9280} \approx \sqrt{9300}$

The square root has two digits.

$\sqrt{93}$ is between 9 and 10 (a little more than half way).
So, a good starting estimate would by 95 or 96 (answers may vary).

9. $\sqrt{22,500} \approx \sqrt{23,000}$
The square root has three digits.
$\sqrt{2} \approx 1.41$
So, a good starting estimate would be 140 or 141 (answers may vary).

10. $\sqrt{30,700} \approx \sqrt{31,000}$
The square root has three digits.
$\sqrt{3} \approx 1.73$
So, a good starting estimate would be 173 (answers may vary).

11. $\sqrt{2750} \approx \sqrt{2800}$
The square root has two digits.
$\sqrt{28}$ is between 5 and 6 but closer to 5.
So, a good starting estimate would be 52 or 53 (answers may vary).

12. $\sqrt{590} \approx \sqrt{600}$
The square root has two digits.
$\sqrt{6} \approx 2.449$
So, a good starting estimate would be 24 or 25 (answers may vary).

13. $\sqrt{51} \approx \sqrt{50}$
The square root has one digit.
$\sqrt{50}$ is between 7 and 8 but much closer to 7.
So, a good starting estimate would be 7 or 7.1 (answers may vary).

14. $\sqrt{3,250,000} \approx \sqrt{3,300,000}$
The square root has four digits.
$\sqrt{3} \approx 1.73$
So, a good starting estimate would be 1730 or a little bit higher (answers may vary).

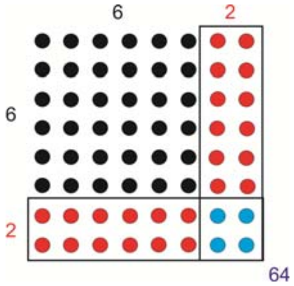
15. $\sqrt{900,000}$
The square root has three digits.
$\sqrt{90}$ is between 9 and 10 (almost exactly halfway).
So, a good starting estimate would be 950 (answers may vary).

The "Binomial Theorem" Algorithm

Let's now explore another way to find square roots by using the binomial theorem. Binomial, from the Latin, literally means "two names" or "two terms" and we will see why in a moment. Let's consider the square of 8 or 8^2. We know that $8^2 = 8 \times 8 = 64$ (and, therefore, $\sqrt{64} = 8$. We now let $8 = 6 + 2$ and substitute. We get:

$$8^2 = (6 + 2)^2 = (6 + 2) \times (6 + 2) = 8 \times 8 = 64$$

Applying the extended distributive property to $(6 + 2) \times (6 + 2)$, we get four products:



Product 1: 6×6
Product 2: 6×2
Product 3: 2×6
Product 4: 2×2

In the figure (depicting area by dots), we can see all four of these products. Notice that the first and last products are squares; i.e., $6 \times 6 = 6^2$ and $2 \times 2 = 2^2$. The second and third terms add (6×2) twice or, by definition of multiplication, $(6 \times 2) + (6 \times 2) = 2 \times (6 \times 2)$ or "twice the product of 6 and 2." Our answer is:

$$6^2 + 2 \times (6 \times 2) + 2^2 = 36 + 24 + 4 = 64$$

Investigate the figure again. Make sure that you see the four parts: $6^2 = 36$, $6 \times 2 = 12$, $2 \times 6 =12$, and $2^2 = 4$. This problem is an elementary example of another law, one that you will learn in later courses (algebra) as the *Binomial theorem*.[5]

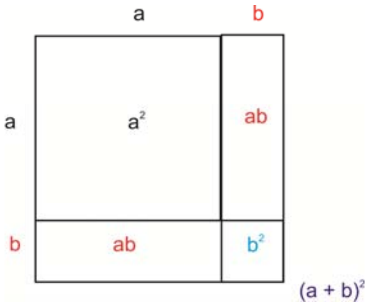
Let' apply this theorem by calculating 41^2. We know that $41 = 40 + 1$. Applying the theorem, we get:

$$41^2 = 41 \times 41 = (40 + 1) \times (40 + 1) =$$
$$1600 + 2 \times (40 \times 1) + 1 =$$
$$1600 + 80 + 1 = 1681$$

Hence, the Binomial theorem is a convenient short-cut multiplication tool!

In general, if *a* and *b* are any two numbers, then (the figure illustrates the areas of the four products):

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$



Note, in the syntax of algebra, 2ab means 2 times a times b. We are going to apply this theorem to generate an algorithm for finding the square root of any number. First, let's explain some principles that "drive" the method.

Principle 1: If *a* consists of *tens* and *b* consists of *units*, then the square of the number a + b is equal to the sum of the squares of the tens (a^2) and units (b^2) plus twice their product (2ab). As we have seen, $41^2 = 41 \times 41 = (40 + 1) \times (40 + 1) = 1600 + 2 \times (40 \times 1) + 1 = 1600 + 80 + 1 = 1681$

Recall that a perfect square is a number which has an *exact* square root. We can develop three more principles based upon this observation.

Principle 2: The square of a single digit number contains no digit of a higher order than tens (e.g., $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, and $9^2 = 81$).

---

[5] *Theorem*, from Greek, means "to look into" or "an insight."

Principle 3: The square of a double digit number contains no significant digit of a lower order than hundreds, nor of a higher order than thousands (e.g., $10^2 = 100$, $99^2 = 9801$).

Principle 4: The square of a number will contain either twice as many digits as the number or twice as many less one.

Thus, we observe:

1^2	=	1 (1 digit or 2×1 = 2 – 1)	10^2	=	100 (3 digits or 2×2 = 4 – 1)
9^2	=	81 (2 digits; 2×1 = 2)	100^2	=	10,000 (5 digits or 3×2 = 6 – 1)
99^2	=	9801 (4 digits; 2×2 = 4)	1000^2	=	1,000,000 (7 digits; 4×2 = 8 – 1)

Hence, we can establish:

Principle 5: If any perfect square be partitioned into periods of two figures each, beginning with the ones position, the number of periods will be equal to the number of figures in the square root of the number. Also, if the number of digits in the number is *odd*, the left-hand period will contain only one digit.

Example 1: Using these principles, let's see if we can calculate the square root of 4356. First, we can partition this four digit number into two periods of two digits each. We can then conclude that the square root of 4356 will consist of two digits.

$$43|56$$

By Principle 3, 56 *cannot* be a part of the square of tens. Hence, the tens of the square root must be found from the first period, or 43.

The greatest number of *tens* whose square is contained in 4300 is 6. Subtracting 3600, the square of 6 tens (60), from 4356, our difference is 756.

$$\begin{array}{rl} 43|56 & \\ 60^2 = 36\ \ 00 & \sqrt{4356} = 6? \\ \cancel{1}756 & \\ 756 & \end{array}$$

By Principle 1, this remainder (756) is composed of *twice* the product of the tens digit (6) by the ones digit (we let *b* = ones digit), and the square of the ones digit; i.e., $756 = 2(60)b + b^2 = 120b + b^2$ for some number *b*. So far, our binomial theorem states:

$4356 = 60^2 + 2(60)b + b^2$
$4356 = 3600 + 120b + b^2$
$756 = 120b + b^2$

We have to find *b*, the ones digit of the square root. Since the product of tens by ones cannot be of a lower order than tens, the last digit 6 in 4356 cannot be part of twice the product of the tens by the ones. This "double product" must therefore be found in 750 (756 – 6).

We take special note of the *double* of the tens of the square root we have found so far (60 × 2 = 120). If we divide 750 by 120, we get $6\frac{1}{4} = 6.25$. The whole number part of this quotient, i.e., 6, will be the ones digit of the square root, or it may be one more (i.e., 7). This quotient cannot be too small, for 750 is at least equal to twice the product of the tens by the ones. But, it may be too large for 750 because the double product may contain tens arising from the square of the units (Principle 2). Let b = 6. Subtracting $120 \times 6 + 6^2$ from 756, our difference is 0. Hence, 66 is the square root of 4356!

$$\begin{array}{r} 43|56 \\ 60^2 = \underline{36\ \ 00} \\ \cancel{1}756 \end{array} \quad \sqrt{4356} = 66$$

$$\begin{array}{r} 6 \\ 120 + 6 = 126\overline{)756} \end{array}$$

In this example, 120 is a *partial* or *trial divisor,* and 126 is the *exact divisor.*

Note that $66^2 = (60 + 6)(60 + 6) = 60^2 + 2\times6\times60 + 6^2 = 3600 + 720 + 36 = 4356$.

If the root contains more than two digits, it may be found by a similar process, as in the following example, *where it will be seen that the partial divisor at each step is obtained by doubling that part of the root already found.* I hope you can see that students in the past who used this method had to know what they were doing!

Example 2: Find the square root of 186,624.

First, we partition:

$$18|66|24$$

Second, 4 hundreds (400) is our first approximation (4 × 4 < 18 or 40 × 40 < 160,000):

$$\begin{array}{r} 18|66|24 \\ 16\ \ 00\ \ 00 \end{array} \quad \sqrt{186{,}624} = 4??$$

Third, we subtract:

$$\begin{array}{r} 18|66|24 \\ 16\ \ 00\ \ 00 \\ 2\ \ 66\ \ 24 \end{array} \quad \sqrt{186{,}624} = 4??$$

Fourth, since $400^2 = 160{,}000$, then our *trial divisor* is 400 × 2 = 800. Because 26,624/800 = 33.28 and *we want the nearest tens*, we add 30 to 400 × 2 = 30 + 800 = 830. Hence, $\frac{26{,}624}{830}$ = 32 R1724. We write:

$$\begin{array}{r} 18|66|24 \\ 16\ \ 00\ \ 00 \\ 400\times2+30 = 830\overline{)2\ \ 66\ \ 24} \\ \underline{2\ \ 49\ \ 00} \\ \cancel{2}7\ \ 24 \\ 17\ \ 24 \end{array} \quad \sqrt{186{,}624} = 43?$$

Fifth, we double 30 to get 60 and our *trial divisor* of 860. We note that $\frac{1724}{860} \approx 2$ so we add 2 to 30 × 2. This gives us $400 \times 2 + 30 \times 2 + 2 = 862$ and $\frac{1724}{862} = 2$

$$
\begin{array}{rl}
 & 18|66|24 \\
 & 16\ \ 00\ \ 00 \quad \sqrt{186,624} = 432 \\
400\times 2+30=830 & \overline{)2\ \ 66\ \ 24} \\
 & \underline{2\ \ 49\ \ 00} \\
 & \ \ \cancel{2}7\ 24 \\
400\times 2+30\times 2+2=862 & \overline{)17\ 24}
\end{array}
$$

$\sqrt{186,624} = 432$!

Note that $432^2 = (400 + 30 + 2)(400 + 30 + 2)$. This is a trinomial times a trinomial. We can extend the distributive property to generate the nine products that we must sum.[6] We get:

$(400 + 30 + 2)(400 + 30 + 2) =$
$400^2 + 2\times400\times30 + 2\times400\times2 + 2\times30\times2 + 30^2 + 2^2 =$
$160,000 + 24,000 + 1600 + 120 + 900 + 4 = 186,624$

Let's try a few examples. I will show the work and you follow along.

Example 3. Find $\sqrt{1764}$

$$
\begin{array}{rl}
 & 17|64 \\
 & \underline{16\ \ 00} \quad \sqrt{1764} = 4? \\
8 & \overline{)1\ \ 64} \\
 & \ \ \ \ \ 2 \\
82 & \overline{)164} \quad \sqrt{1764} = 42
\end{array}
$$

Example 4. Find $\sqrt{169}$

---

[6] In general, $(a + b + c)(a + b + c) = a^2 + 2ab + 2ac + 2bc + b^2 + c^2$. Each term in the first factor must be multiplied by each term in the second factor. Hence, we have nine products (3 × 3) to compute: $(\overbrace{\overbrace{\overbrace{a+b+c)(a}^{a^2}+b}^{ab}+c}^{ac})$, $(a+\overbrace{\overbrace{\overbrace{b+c)(a}^{ab}+b}^{b^2}+c}^{bc})$, and $(a+b+\overbrace{\overbrace{\overbrace{c)(a}^{ac}+b}^{bc}+c}^{c^2})$.

$$1|69$$

$$\underline{1\ \ 00} \quad \sqrt{169} = 1?$$

$$2\overline{)69}$$

$$\begin{array}{r} 3 \\ 23\overline{)69} \end{array} \quad \sqrt{169} = 13$$

Example 5. Find $\sqrt{2809}$

$$28|09$$

$$\underline{25\ \ 00} \quad \sqrt{2809} = 5?$$

$$10\overline{)3\ \ 09}$$

$$\begin{array}{r} 3 \\ 103\overline{)309} \end{array} \quad \sqrt{2809} = 53$$

Example 6. Find $\sqrt{324}$

$$3|24$$

$$\underline{1\ \ 00} \quad \sqrt{324} = 1?$$

$$2\overline{)2\ \ 24}$$

$$\begin{array}{r} 3 \\ 28\overline{)224} \end{array} \quad \sqrt{324} = 18$$

In this case, $\begin{array}{r} 11 \\ 2\overline{)224} \end{array}$. Since 11 will not work in the units place in our square root, we count backwards. 10 is two digits so it will not work. We try 9 but it is too large (19^2 = 361). Finally, 8 works.

Find the square root of the following numbers (answers are on the next couple of pages).

1. $\sqrt{144}$
2. $\sqrt{1024}$
3. $\sqrt{961}$
4. $\sqrt{2025}$
5. $\sqrt{625}$
6. $\sqrt{576}$
7. $\sqrt{41,616}$

Perfect squares are given in these problems. If the number is not a perfect square, we continue the algorithm passed the decimal point into the tenths, hundredths, thousandths, etc. positions. We stop and round for the precision desired. Remember, students who took arithmetic in the years preceding the 1970s knew how to work this algorithm!

CALCULATING SQUARE ROOTS BY HAND

Answers.

1. $\sqrt{144}$

$1\,|\,44$

$\underline{\ 1\ \ 00\ }\quad \sqrt{144} = 1?$

$2\overline{)44}$

$\begin{array}{r} 2 \\ 22\overline{)44} \end{array}\quad \sqrt{144} = 12$

2. $\sqrt{1024}$

$10\,|\,24$

$\underline{\ \ 9\ \ 00\ }\quad \sqrt{1024} = 3?$

$6\overline{)1\ \ 24}$

$\begin{array}{r} 2 \\ 62\overline{)124} \end{array}\quad \sqrt{1024} = 32$

3. $\sqrt{961}$

$9\,|\,61$

$\underline{\ 9\ \ 00\ }\quad \sqrt{961} = 3?$

$6\overline{)\ 61}$

$\begin{array}{r} 1 \\ 61\overline{)61} \end{array}\quad \sqrt{961} = 31$

4. $\sqrt{2025}$

$20\,|\,25$

$\underline{\ 16\ \ 00\ }\quad \sqrt{2025} = 4?$

$8\overline{)\ 4\ \ 25}$

$\begin{array}{r} 5 \\ 85\overline{)425} \end{array}\quad \sqrt{2025} = 45$

5. $\sqrt{625}$

$6\,|\,25$

$\underline{\ 4\ \ 00\ }\quad \sqrt{625} = 2?$

$4\overline{)2\ \ 25}$

$\begin{array}{r} 5 \\ 45\overline{)225} \end{array}\quad \sqrt{625} = 25$

6. $\sqrt{576}$

$$\begin{array}{l} 5\,|\,76 \\ \underline{4\ \ 00} \qquad \sqrt{576} = 2? \\ 4\overline{)1\ 76} \end{array}$$

$$\begin{array}{l} 4 \\ 44\overline{)176} \qquad \sqrt{576} = 24 \end{array}$$

7. $\sqrt{41,616}$

$$\begin{array}{l} 4\,|\,16\,|\,16 \\ 4 \qquad\qquad \sqrt{41,616} = 2?? \\ 400\overline{)1616} \qquad \sqrt{41,616} = 20? \end{array}$$

$$\begin{array}{l} 4 \\ 404\overline{)1616} \qquad \sqrt{41,616} = 204 \end{array}$$