THE WONDER OF THE PYTHAGOREAN MEANS

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THE NUMERICAL AVERAGE

A primary measure of statistics is the mean, the average, or, to be mathematically precise, the arithmetic mean. The arithmetic mean, μ, of two positive numbers, *a* and *b*, is:

$$\mu = \frac{a+b}{2}$$

GEOMETRIC REVELATIONS

In geometry, we encounter the geometric mean. Given three positive numbers *a*, *b*, and *c*, *b* is the geometric mean if this proportion holds:

$$\frac{a}{b} = \frac{b}{c}$$

Using Algebra, we can solve for *b*. We get:

$$\frac{a}{b} = \frac{b}{c} \Leftrightarrow b^2 = ac \Rightarrow b = \sqrt{ac}$$

What is so "geometric" about the geometric mean? In a right triangle, the altitude dropped from the vertex of the right angle is the geometric mean between the two segments into which the base of the altitude divides the hypotenuse. In the figure, $\frac{AC}{PC} = \frac{PC}{CB}$ or $\frac{a}{x} = \frac{x}{b}$. From this proportion, solving for *x*, we get:

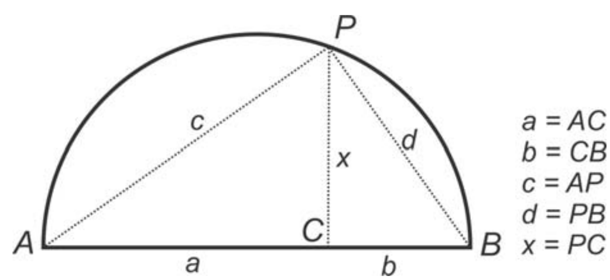

Figure 1

$x = \sqrt{ab}$ (for the proof, see Euclid's Elements or any good High School Geometry text).

MUSICAL HARMONY

The arithmetic mean and geometric mean are two of the three famous Pythagorean means. Since Pythagoras (6th century BC) connected plucked strings to mathematics, the third mean of this trio is related to music. It is called the harmonic mean. The harmonic mean, *h*, of two positive numbers *a* and *b* is defined as follows:

$$h = \frac{2}{a+b/_{ab}} = 2\left(\frac{ab}{a+b}\right) = \frac{2ab}{a+b}$$

First, note carefully how the arithmetic mean is embedded of this definition. Second, what is so "harmonic" about the harmonic mean? What mathematicians called the harmonic sequence[1] plays a crucial role in music theory for the fundamental musical intervals are derived from numbers in this sequence. Plucking a string of certain length produces a note of a particular frequency (i.e., vibrations per second).

[1] A sequence, in mathematics, is a list of numbers that follow a given pattern.

When a string is depressed at half its length, its octave is produced which vibrates at twice the frequency. The sequence of depression of strings produces the harmonic sequence in the table below:

Table 1

Note	*n*	Length of String	Frequency
cc	1	1	64
c	2	$\frac{1}{2}$	128
g	3	$\frac{1}{3}$	192
c′	4	$\frac{1}{4}$	256
e	5	$\frac{1}{5}$	320
g	6	$\frac{1}{6}$	384
b flat	7	$\frac{1}{7}$	448
c″	8	$\frac{1}{8}$	512
d	9	$\frac{1}{9}$	576
e	10	$\frac{1}{10}$	640
f sharp	11	$\frac{1}{11}$	704
g	12	$\frac{1}{12}$	768
a	13	$\frac{1}{13}$	832
b flat	14	$\frac{1}{14}$	896
b natural	15	$\frac{1}{15}$	960
c‴	16	$\frac{1}{16}$	1024

Some of the intervals between notes in this scale are:

Table 2

Interval	Name
1 to 2	octave
2 to 3	fifth
3 to 4	fourth
4 to 5	major third
5 to 6	minor third
8 to 9	second
9 to 10	second
15 to 16	semitone

Note especially that the harmonic sequence, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, … is the reciprocal of the sequence of natural numbers; i.e., 1, 2, 3, 4, …!

In the sequence of natural numbers, each term of the sequence is the *arithmetic mean* of the two terms immediately preceding and following it. Recalling that $\mu = \frac{a+b}{2}$, the arithmetic mean of 2 and 4 is 3.

$$\frac{2+4}{2} = \frac{6}{2} = 3$$

In similar fashion, each term of the sequence, 1, $\frac{1}{2}$, $\frac{1}{3}$, … is the *harmonic mean* of the two terms immediately preceding and following it. Recalling that $h = \frac{2ab}{a+b}$, the harmonic mean of $\frac{1}{2}$ and $\frac{1}{4}$ is $\frac{1}{3}$.

$$\frac{2\left(\frac{1}{2}\times\frac{1}{4}\right)}{\frac{1}{2}+\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{2}{4}+\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4}\div\frac{3}{4} = \frac{1}{\cancel{4}_1}\times\frac{\cancel{4}^1}{3} = \frac{1}{3}$$

Let's explore more wonders. Given the arithmetic mean $\mu = \frac{a+b}{2}$, we know that the reciprocal of a is $1/a$ and the reciprocal of b is $1/b$. Taking the arithmetic mean of these reciprocals, we get:

$$\frac{\frac{1}{a}+\frac{1}{b}}{2} = \frac{\frac{a+b}{ab}}{2} = \frac{a+b}{2ab}$$

Taking the reciprocal of the arithmetic mean of the reciprocals of a and b (that is a "mouth" full, isn't it?) gives us the harmonic mean!

$$h = \frac{2ab}{a+b}$$

THE ALGEBRA OF THE THREE MEANS

Is there an algebraic connection between the Pythagorean means? Given two positive numbers a and b, where μ = arithmetic mean, G = geometric mean, and h = harmonic mean, we have these relationships:

$$\mu = \frac{a+b}{2}$$

$$G = \sqrt{ab}$$

$$h = \frac{2ab}{a+b}$$

Let's do some Algebra!

$$G = \sqrt{ab} \Rightarrow G^2 = ab \Leftrightarrow 2G^2 = 2ab$$

$$\mu = \frac{a+b}{2} \Leftrightarrow 2\mu = a + b$$

Since $h = \frac{2ab}{a+b}$, then, by substitution,

$$h = \frac{2G^2}{2\mu} = \frac{G^2}{\mu}$$

$$h = \frac{G^2}{\mu} \Leftrightarrow h\mu = G^2 \Rightarrow G = \sqrt{h\mu}$$

Also, $h\mu = G^2 \Leftrightarrow \mu = \frac{G^2}{h}$

> A peculiar beauty reigns in the realm of mathematics, a beauty which resembles not so much the beauty of art as the beauty of nature and which affects the reflective mind, which has acquired an appreciation of it, very much like the latter.
>
> E. E. Kummer, *Berliner Monatsberichte* [1867], p. 395.

In summary, we have established the algebraic harmony between the three Pythagorean means by these three formulas:

1. $h = \frac{G^2}{\mu}$
2. $G = \sqrt{h\mu}$
3. $\mu = \frac{G^2}{h}$

Can we see these relationships geometrically? Yes! I leave it to the reader to derive the proof.

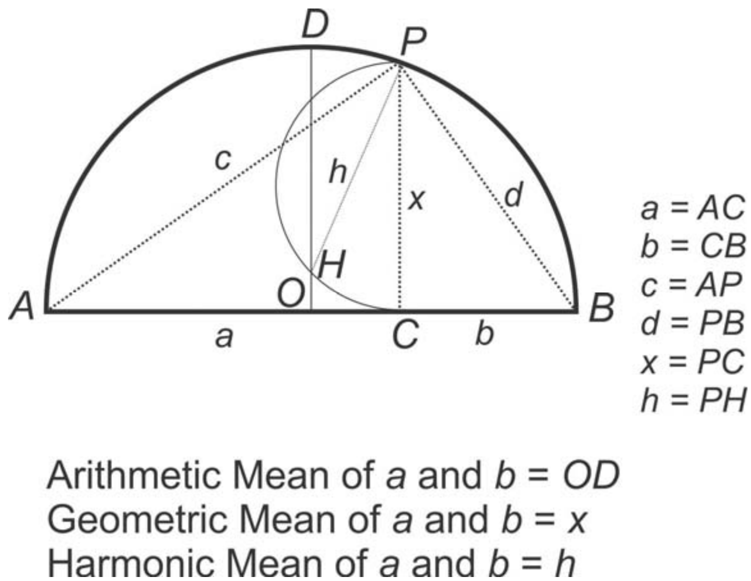

Figure 2

INTERLACING THREADS

Francis Schaeffer (1912-1984) once said, "We tend to study all our disciplines in unrelated parallel lines … We have studied exegesis as exegesis, our theology as theology, our philosophy as philosophy; we study something about art as art; we study music as music, without understanding that these are things of man, and the things of man are never unrelated parallel lines."[2]

> The works of God are pleasant (Genesis 2:9). The beauty of God's works gives pleasure to the senses (both cognitive and tactile).

My goal in this study is to demonstrate how one can see the interlacing threads, Schaeffer's related parallel lines, embedded in mathematical propositions, to discern the harmonious relationships not only within mathematics but between mathematics and the world, God's world, that we experience.

We have seen that the Pythagorean means inhere in each other, how they reveal a harmony both algebraically and geometrically, cognitive and tactile.

These means are not only connected statistics, geometry, and music. The geometric mean has been useful in describing proportional growth,

[2] Francis Schaeffer, *Francis A. Schaeffer: Trilogy* (Wheaton: Crossway Books, 1990), p. 212.

both constant and varying, either in the growth of money or crop yield. In the business world, the geometric mean of growth rates is known as the compound annual growth rate (CAGR).

The harmonic mean is revealed in numerous ways and serves as a beautiful example of the principle of unity in diversity.

In trigonometry, given an angle θ where tan θ = a/b, we can calculate tan 2θ (double angle), instead of using the traditional double angle formula, as follows:

$$\tan 2\theta = \left(\frac{2ab}{a+b}\right)\left(\frac{1}{b-a}\right)$$

In physics, averages involving rates and ratios are best given using the harmonic mean. Examples involve objects working in parallel; e.g., draining water simultaneously with two pumps and the effective resistance of two or more electrical resistors wired in parallel. Space does not allow for detailed explanations of the harmonic mean applied to computer science, hydrology, population genetics, fuel economy, and price/earnings ratios.

Series

I close with a fascinating analysis of the harmonic series, investigated by the medieval bishop and mathematician Nicole Oresme (ca. 1323-1382). In mathematics, a series is the sum of a sequence. The most common types of series are classified arithmetic or geometric. There can be a finite or an infinite number of terms in a series. It is when the number of terms is infinite that the imagination is intrigued and challenged.

Arithmetic Series

In an arithmetic series, the difference between any two successive terms is constant. In symbols, where b represents that difference, the arithmetic series looks like this:

$$a + (a + b) + (a + 2b) + (a + 3b) + \ldots + [a + (n - 1)b]$$

Here is an example of an infinite arithmetic series where $a = 1$, $b = 2$:

$$1 + 3 + 5 + 7 + 9 + \ldots$$

Note: 3 is the arithmetic mean of 1 and 5, 5 is the arithmetic mean of 3 and 7, etc. This series is so named because the arithmetic mean is embedded in it.

Geometric Series

In a geometric series, the ratio of any two consecutive terms is a constant. In symbols, where r represents that ratio, the geometric series looks like this:

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1}$$

The following is an example of an infinite geometric series where we calculate the next term by dividing the previous term by 10. In this example, $a = 1$, $b = \frac{1}{10}$:

$$1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ldots$$

Note: $\frac{1}{10}$ is the geometric mean of 1 and $\frac{1}{100}$, $\frac{1}{100}$ is the geometric mean of $\frac{1}{10}$ and $\frac{1}{1000}$, etc. This series is so named because the geometric mean is embedded in it.

We can find the sum of this infinite series by using a formula derived from mathematical analysis. I will leave it to the reader to discover this formula:

$$1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ldots=1\frac{1}{9}$$

Here we have the sum of an infinite geometric series *converging* to a particular number. Every infinite geometric series does not converge in this manner, though. Consider this geometric series where we calculate the next term by multiplying the previous term by 10:

1 + 10 + 100 + 1000 + 10,000 + …

This pile keeps getting larger and larger. Its sum *tends* to infinity and because of this, it is called a *divergent* series.

The next example of an infinite geometric series, where we calculate the next term by multiplying the previous term by 1, also *tends* to infinity as a limit. For the millionth term, the sum is 1,000,000.

1 + 1 + 1 + 1 + …

This geometric series, where we calculate the next term by multiplying the previous term by -1, is troublesome:

1 – 1 + 1 – 1 + 1 – 1 + …

After two terms, the sum is 0, after three the sum is 1, after four the sum returns to 0. So, the partial sums oscillate about 0 and 1. It's like the waitress who asks Mr. Swing, "What will you have? Coffee or tea?" Mr. Swing answers, "Yes." Because of this oscillation, this series does not have a limiting value. It is called an indefinite or indefinitely divergent series.

Harmonic Series

Let's return to Oresme's work. We can find the sum this infinite series:

Series 1:

$$1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ldots=1\frac{1}{9}$$

We note that each succeeding term gets smaller and smaller. At infinity, the terms approach 0. If each succeeding term gets smaller and smaller, can we *always* conclude that the sum of the infinite series converges? The answer to this question shaped the framework of the development of the theory undergirding differential and integral calculus, a mathematical method that is the ground of the success of Western technology.

As we have seen, the harmonic sequence is:

$$1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\cdots$$

So, we can write the harmonic series as follows:

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ (N.B.: This series is *not* a geometric series!)

Oresme wanted to find the sum this series. We can see right away that each succeeding term gets smaller and smaller, and we can conclude that at infinity the terms approach 0. But, these terms approach infinity at a much slower rate than Series 1. In Series 1, our terms gallop toward zero. In this sequence, the terms walk leisurely toward zero. What is the sum of this series?

Series 2:

$$1 + \frac{1}{2} + \underline{\frac{1}{3} + \frac{1}{4}} + \underline{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \underline{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}} + \ldots$$

Oresme developed the following investigation of the partial sums of this series. Starting with the third term, we add blocks of terms containing 2, 4, 8, 16, … terms, etc., noting the following relationships:

Table 3

Terms summed	Sum
1st	1
2nd	$\frac{1}{2} = \frac{1}{2}$
3rd and 4th	$\frac{1}{3} + \frac{1}{4} > 2\left(\frac{1}{4}\right) = \frac{1}{2}$
5th to 8th	$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 4\left(\frac{1}{8}\right) = \frac{4}{8} = \frac{1}{2}$
9th to 16th	$\frac{1}{9} + \ldots + \frac{1}{16} > 8\left(\frac{1}{16}\right) = \frac{8}{16} = \frac{1}{2}$
17th to 32nd	$\frac{1}{17} + \ldots + \frac{1}{32} > 16\left(\frac{1}{32}\right) = \frac{16}{32} = \frac{1}{2}$

What is going on? First, if you have been led to think that anything "medieval" is equivalent to ignorance and stupefaction, then think again. Oresme's approach is nothing less than brilliant. Second, Oresme made use of the following observation: when comparing two fractions with the same numerator (e.g., $\frac{1}{3}$ and $\frac{1}{4}$), the fraction with the greater denominator is smaller; i.e., $\frac{1}{4} < \frac{1}{3}$ or if we cut a pie into more slices, each slice will be smaller.

Given this understanding, consider the first partial sum $\left(\frac{1}{3} + \frac{1}{4}\right)$. If we compare this sum to $\left(\frac{1}{4} + \frac{1}{4}\right)$, we know that the second sum is smaller than the first sum. Or, the first sum is greater than the second sum.

Consider the next partial sum $\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)$. If we compare this sum to $\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)$, we know that the first sum is greater than the second sum. We can continue this comparison with the succeeding partial sums. We have constructed a new series that looks like this:

Series 3:

$1+\frac{1}{2}+\underline{\frac{1}{4}+\frac{1}{4}}+\underline{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}+\underline{\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}}+\ldots$ The value of the partial sums of this series is, one after another:

$$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$
$$\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$
$$\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{8}{16}=\frac{1}{2}$$
etc.

The sum of every group equals $\frac{1}{2}$. If we construct 10 of these groups, the sum would be $10 \times \frac{1}{2} = 5$. If we construct 100 groups, the sum is 50. The sum of 1000 groups is 500. The sum of 1,000,000 groups is 500,000. The partial sums get larger and larger as we walk leisurely toward infinity.

Since the sum of Series 3 *tends* to infinity, then so does the sum of Series 2. Why? In Series 2 the sums of the partial groups *are always larger* than the sum of the corresponding partial groups in Series 3.

Series 2:

$$1+\frac{1}{2}+\underline{\frac{1}{3}+\frac{1}{4}}+\underline{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}+\underline{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}+\ldots$$

The sum of Series 2 *tends* to infinity. It is a *divergent* and mathematicians have denoted this type of series, because of its musical connections, as *harmonic*.

Our conclusion to this excursion into all things harmonic is two-fold. First, for an infinite series to converge, its terms must tend toward zero at "battle speed." If the terms in a series tend toward zero at "cruising speed," then the sum probably diverges. Second, starting from the study of music by Pythagoras, we end up with the analysis of infinite processes embedded in convergent and divergent series. Who would have thought of this at the beginning? The analysis of convergent and divergent series is critical in the study of higher mathematics, particularly calculus, and the revelation of such mathematics to our experiences in God's world.

Such is the wonder of mathematics!