THE MATHEMATICAL UNIVERSE

THE ORDERLINESS OF NATURE CAN BE EXPRESSED MATHEMATICALLY. WHY?

BY JOHN D. BARROW

(From The World & I, 1989, p. 307-311)

A Yale professor was once asked to settle a dispute between two rival factions in the university: 'What was more important, teaching languages or teaching mathematics? "Mathematics *is* a language" was his answer.

In the world of science, mathematics indeed seems to be the language that allows us to talk most effectively and logically about the nature of things. But mathematics differs from other languages like English or Spanish. Indeed, it possesses a built-in logic, and is thus more akin to a computer language. When we write a grammatically correct English sentence like "All dogs have four legs and my table has four legs, so my table is a dog," there is no guarantee the sentence will be logically correct or correspond with events in the world. Conversely, the grammatical incorrectness of a phrase like "to boldly go where no man have gone before" does not render its realization impossible. One can break a rule of English grammar without falling into meaninglessness, but break a rule of mathematics and disaster ensues. If one false mathematical statement is allowed, it can be used to prove the validity of *any* mathematical statement. When Bertrand Russell once made this claim during a lecture, he was challenged by a skeptical heckler to prove that the questioner was the pope if twice 2 were 5. Russell at once replied: "If twice 2 is 5, then 4 is 5; subtract 3, then 1 equals 2. But you and the pope are 2; therefore, you and the pope are 1!"

So mathematics is a language with a built-in logic. But what is so striking about this language is that it seems to describe how the world works—not just sometimes, not just approximately, but invariably and with unfailing accuracy. All the fundamental sciences—physics, chemistry, and astronomy—are mathematical sciences. No phenomenon has ever been discovered in these subjects for which a mathematical description is not only possible but also beautifully appropriate. Yet one could still fail to be impressed. After the fact, perhaps, we can force any hand into some glove, and maybe we have chosen to pick the mathematical glove because it is the only one available. It is striking, however, that physicists so often find that some esoteric mathematical structure, invented by mathematicians in the dim and distant past only for the sake of its elegance and curiosity value, is precisely what is required to make sense of new observations of the world. In fact, confidence in mathematics has grown to such an extent that one now expects (and finds) interesting mathematical structures to be deployed in nature. Scientists look no further when they have found a mathematical explanation.

There are many striking examples of the unexpected and curious effectiveness of mathematics. In 1914, when Einstein was struggling to formulate a new description of gravity to supersede that of Newton, he wished to endow the universe with curved space and time, and to codify the laws of nature in a manner that would apply for any observers no matter what their state of motion. His old student friend, the mathematician Marcel Grossman, introduced him to a little-known branch of nineteenth-century mathematics, called tensor calculus, that was tailor-made for his purposes. Upon adopting this mathematical language, how Einstein would describe laws of nature became clear and (if one is that clever) obvious.

In modern times, particle physicists have discovered that symmetry dictates the way elementary particles behave. Particular collections of related particles can behave in any way they choose so long as a particular abstract pattern is preserved. The laws of nature are superficially the catalog of habitual things that occur in the world while yet preserving these patterns. With every such catalog of changes one can always find an unchanging pattern, though the pattern is often subtle and rather abstract.

In the last century, mathematicians set about investigating all the possible patterns that one could invent. These patterns are the subject matter of a branch of mathematics called *group theory.* The catalog of mathematical patterns that the early group theorists established has become the guiding force in the study of elementary particles. So successful were these simple patterns in describing the way the forces and particles of nature behave that physicists have taken to exploring all the possible patterns to discover those that give the most interesting insight into the origin of the universe. In this way the expectation that elegant mathematical ideas will be found in nature allows detailed predictions to be made about the behavior of elementary particles. Experiments with particle accelerators enable these aesthetic fancies to be tested against the facts.

The search for the "theory of everything" has sought the most elegant and all-encompassing pattern into which all the other patterns can be successfully fitted. The theory of *superstrings* (in which the most basic element of nature is not "point-like" but line-like") has become a focal point of interest in such a search and has also created a new phenomenon. Physicists have encountered concepts that require the invention of new mathematical structures and ideas. For the first time, off-the-shelf mathematics is not enough to unravel all the patterns.

Elsewhere, the development of fractal geometry has given insight into the patterns within a whole spectrum of natural phenomena—from the clustering of galaxies to the structure of snowflakes. Many superficially chaotic situations possess a deep, underlying mathematical order. Again, fractals were once no more than an obscure branch of mathematics investigated for its own sake.

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Scientists seem to believe so deeply in the mathematical structure of nature that it is an unquestioned article of faith that mathematics is both necessary and sufficient to describe everything from the inner space of elementary particles to the outer space of stars and galaxies—even the universe itself. What are we to make of the ubiquity of mathematics in the constitution of the universe? Is it evidence of a deep logic within the universe? If so, where does that logic come from? Is it just a creation of our own minds, or is God a mathematician?

Why does reality follow a mathematical lead? The answers to such puzzling questions depend crucially upon what we think mathematics actually is. There are four clear options—formalism, *inventionism, realism,* and *constructivism.*

FORMALISM

At the beginning of this century, mathematicians faced several bewildering problems. Logical paradoxes like that of the barber ("A barber shaves only those individuals who do not shave themselves. Who shaves the barber?") or the problem with sets of sets ("Consider the collection of all collections: Is it a member of itself?") seemed to undermine the entire mathematical edifice. Who knows where the next paradox might surface? In the face of such difficulties David Hilbert, the foremost mathematician of the day, that we cease worrying about the *meaning* of mathematics altogether. Instead, he offered, *define* mathematics to be no more and no less than the tapestry of formulas that can be created from any set of initial axioms by manipulating the symbols involved according to specified rules. This procedure, it was believed, could not create paradoxes. The vast embroidery of interwoven logical connections that results from the manipulation of all the possible starting axioms according to all the possible noncontradictory rules fixes, according to Hilbert, what mathematics "is."

Clearly, for Hilbert and his disciples the miraculous applicability of mathematics to nature is something about which they neither care nor seek to explain. Mathematics, for them, does not have any *meaning.* The axioms and rules for the manipulation of symbols are not connected with observed reality in any necessary way. Formulas exist on pieces of paper, but mathematical entities have no other claim to existence. The formalist would no more offer an explanation for the mathematical character of physics than would he seek to explain why physical phenomena do not obey the rules of poker or blackjack.

Hilbert thought that this strategy would create a body of logically valid reasoning and result, free of all possible paradoxes, which would be defined to be mathematics. Given any mathematical statement, it would be possible in principle to determine whether it was a true or false conclusion from any particular set of starting assumptions by working through the logical network of connections. Hilbert and his disciples set to work, confident that they could encompass all truth within this straitjacket.

Unfortunately, and totally unexpectedly, the enterprise collapsed overnight. In 1931, Kurt Gödel, an unknown young mathematician at the University of Vienna, showed that Hilbert's goal was unattainable. Whatever set of consistent axioms one chooses, whatever set of consistent rules one adopts for manipulating the mathematical symbols involved, there must always exist some statement, even though framed in the language of those symbols, whose truth or falsity cannot be decided using those axioms and rules. Mathematical truth is something larger than axioms and rules. Try solving the problem by adding a new rule or a new axiom and one merely creates new undecidable statements. Checkmate. Hilbert's program cannot work. If you want to understand mathematics fully you have to go outside of mathematics. Incidentally, if a "religion" is a system of ideas that contains unprovable statements, then Gödel has taught us that not only is mathematics a religion, but it is the only religion that can prove itself to be one!

INVENTIONISM

Inventionism is the belief that mathematics is simply what mathematicians do. We invent mathematics: We do not discover it. Mathematical entities like sets or triangles would not exist if there were no mathematicians. The inventionist is not very impressed by the effectiveness of describing the world by mathematics. The reason we find mathematics so useful is perhaps merely an indication of how little is known of the physical world. It is only the properties well-suited to mathematical description that we have been able to uncover. So, although we "see" the universe to be mathematical, this does not mean that it really *is* mathematical any more than the sky is pink because it looks that way when we wear rose-colored spectacles.

If, however, mathematics were entirely a human invention and used by scientists simply because it is useful and available, then one might expect significant cultural differences within the subject. But whereas there are discernible styles in the presentation of mathematics and in the type of mathematics investigated in different cultures, this diversity is just a veneer. The independent discovery of the same mathematical theorems by different mathematicians from totally different economic, cultural, and political backgrounds at different times throughout history argues against such a simple view. Moreover, the phenomenon of the independent, multiple "invention" of the same mathematical truth sets mathematics apart from music or the arts. Pythagoras' theorem was discovered many times by different thinkers. It is inconceivable that Shakespeare's *Hamlet* or Beethoven's Fifth Symphony could be independently recreated. This contrast argues that the foundation of mathematics lies outside of the human mind and is not totally fashioned by our human way of thinking.

Kant's categories of thought need not place a significant barrier between our understanding of the world and the bedrock of reality. For our minds and their categories of thinking are the products of the evolutionary process. Like our other bodily features, they are the products of a natural selection process in which reality (not perceived reality) dictates what survives. Accordingly, our eyes have evolved as effective light detectors in tune with the real properties of light. Our eyes carry information about the real nature of light, our ears carry information about the reality of sound. Likewise, our minds should be clear recorders of those aspects of reality that are crucial for the evolution of sentience. If our mental categories were distorting reality, we would not have survived. Certain basic mathematical notions like symmetry, geometry, and counting could thus have emerged as innate concepts preexisting in the human mind.

However, even if we believe this argument, we still have to consider those esoteric areas of science and mathematics that could have played no role in our evolution. Moreover, mathematics uses all manner of concepts that we do not directly experience in reality - irrational numbers, points, or spaces with more than three dimensions. Nevertheless, although we may have invented some mathematics for our own special purposes, the idea that we have invented it all—so that it does seem to describe observed reality—seems farfetched because we can so often use this same mathematics to predict the existence of new and unsuspected physical phenomena.

REALISM

The most straightforward view of mathematics is to maintain that the world *is* mathematical in some deep sense. Mathematical concepts exist, and they are discovered by mathematicians, not invented. Mathematics exists whether or not there are mathematicians. It is a universal language that could be used to communicate with alien beings who have developed independently from ourselves. For the realist, the number seven exists as an immaterial idea that we see realized in specific cases, like seven dwarfs, seven brides, or seven brothers. It is sometimes called *mathematical Platonism,* because it assumes that there exists some other world of perfect mathematical forms that are the blueprints from which our particular experience is derived. Realism of this sort seems tantamount to the view that God is a mathematician. And indeed, if the entire material universe is described by mathematics (as modern cosmology leads us to expect), then there must exist some immaterial logic that transcends and permeates the material universe.

Realism regards the surprising effectiveness of mathematics in describing nature as crucial evidence that supports a realist approach. Most scientists and mathematicians carry out their work as if realism were true, even though they might be loath to defend it too strongly outside the workplace. But realism of this sort has a most extraordinary consequence. One might imagine writing a computer program to simulate some complex physical process, like the formation of a solar system or the division of a cell. In principle, a program complicated enough to simulate the whole universe and all the processes that go on in it could exist. Within such a program, the formation of galaxies and stars would be simulated together with the biological evolution of animals and human beings.

If the simulation were perfect, then the simulated people would have simulated thoughts and make simulated observations about other things and people in the program. They would regard all these things to be as real as real could be. They could not tell from "inside" the computer program that they were on the inside. Being part of the software, they could not tell what sort of hardware they were being run on (or even if there existed any hardware). Similarly, we cannot tell whether or not we are part of some Superbeing's computer simulation.

We might wonder if it is possible, in fact, to write a computer program to simulate everything that happens in nature. This question leads us to consider the last of our options regarding the interpretation of mathematics.

CONSTRUCTIVISM

The last of our "isms" was another response to the uncertainty about logical paradoxes that spawned formalism in the early years of this century. The constructivists' starting point, according to Leopold Kronecker, one of its creators, is the assertion that "God made the integers; all else is the work of man." What he meant was that we should accept only the simplest possible mathematical notions as a starting point, and we should derive everything else from these intuitively obvious notions. By this conservative stance the constructivists hoped to avoid encountering or manipulating concepts—like "infinity"—about which we could have no concrete experience and which have counterintuitive properties (infinity minus infinity equals infinity, for instance).

Mathematics consists, then, of the collection of statements that can be constructed in a finite number of deductive steps, taking whole numbers as the starting point. On this view, the "meaning" of a mathematical formula is simply the finite chain of computations that have been used to reach it.

This view may sound harmless enough, but it has dire consequences. It creates a new category for mathematical statements, which can now be true, false, or *undecided.* A statement whose truth cannot be decided in a *finite* number of constructive steps is given this last limbo status. The most important consequence of this policy is that a statement is no longer *either* true *or* false. This is reminiscent of Scottish courts of law, where a verdict of guilty, not guilty, or *not proven* may be returned (the latter permits a retrial of the defendant on the same charge), whereas English or American courts require a verdict of either guilty or not guilty.

Traditionally mathematicians have developed many ways of proving formulas to be true, methods that do not correspond to a finite number of constructive steps. One famous method beloved of the ancient Greeks is called *reductio ad absurdum.* If we want to show something to be false, we assume it to be true and from that assumption deduce something contradictory (like 2 equals 1). From this contradiction we conclude that our original statement could not have been true. This argument is based on the presumption that a statement that is not true is false, an invalid strategy, according to the constructivists' rules.

Furthermore, constructivism outlaws the whole body of mathematical theorems that prove that something exists but which do not construct an example of that something. This philosophy would have interesting consequences if adopted in physics because many important physical theories, like Einstein's general relativity or Niels Bohr's quantum mechanics, make important use of nonconstructive reasoning in deducing properties of the universe. To most mathematicians, a constructivist approach seems rather depressing—tantamount to fighting with one arm tied behind your back.

Now what has constructivism to say about the mathematical character of nature? We can see that constructivism latches on to what remained of Hilbert's formalistic program following Gödel's devastating discovery. There must always be some statements whose truth can neither be proved nor disproved, but what about all those statements whose truth can be determined by any of the traditional methods of mathematics? How many of them could the constructivists prove? Can we build, at least in principle, a computer that reads input, displays the current state of the machine, and possesses a processor for determining a new state from its present one, and then use it to decide whether a given statement is true or false after a finite time? Is there a specification for a machine that can decide for us whether all the decidable statements of mathematics are either true or false? Contrary to the expectations of many mathematicians, the answer was no. Alan Turing at Cambridge, and Emil Post and Alonzo Church at Princeton, showed that there are statements whose truth would require an infinite time to decide. These statements are, in effect, infinitely deeper than the logic of step-by-step computation. The idealized computer we have described is called a *Turing machine;* it is the essence of every computer. No real computer possesses greater problem-solving ability.

The mathematical operations that a Turing machine cannot perform in finite time are called *non- computable functions.* They could have interesting consequences. If, for example, the action of the human mind involves noncomputable operations, then the quest for artificial intelligence cannot succeed in producing computer hardware able to mimic the complexity of human consciousness.

If we return to the puzzle of the applicability of mathematics to nature, we can cast the question into an interesting statement about computability. If an operation is computable, this means we can fabricate a device from matter whose behavior mimics that operation. Typical devices might be swinging pendulums or electrical impulses. Conversely, physical devices like these can be well described by computable mathematical operations. The fact that is well described by mathematics is equivalent to fact that the simplest mathematical operations—like addition and multiplication—and other, more complicated operations used so effectively in science are computable functions. If they were not, then they could not be equivalent to any natural process, and we would not be terribly impressed by the usefulness of mathematics.

It is fascinating to ask whether the laws of nature contain noncomputable elements. Already, the quest to find a quantum theory of the whole universe has created this possibility. If natural laws do contain noncomputable elements, then their consequences cannot be discovered by any systematic calculation that merely applies the same principles over and over again; each step requires that qualitatively different and novel principles be used. Whereas some sequences are simple in the sense that a brief formula can be given to generate the sequence, others are so complex that no abbreviation of the sequence can contain enough information to generate it. We might expect the universe to be such an entity: one ultimately irreducible to any abbreviated formula and defined most simply by nothing less than its own unfolding sequence of events.

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The universe may or may not be intrinsically mathematical. If it is not, then we are extremely fortunate in that mathematical language works in areas it was never designed to cover. If the universe *is* mathematical in some deep sense, then the undecidabilities demonstrated by Gödel and Turing are part of the fabric of the universe rather than merely products of our minds. They show that even a mathematical universe is more than axioms, more than computation, more than logic—and more than mathematicians can know.

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