

# Proof that $\sqrt{2}$ is irrational

by James D. Nickel

When we convert fractions into decimals, we notice two things. Either the decimals (1) terminate or they (2) repeat in a recognizable pattern. For example,  $\frac{3}{4} = 0.75$  (decimal expansion terminates at 5) and

$\frac{1}{7} = 0.\overline{142857}$  where 142857 is the repetend (digits that repeat, *ad infinitum*).

It is also possible to construct a fraction given its decimal expansion (whether terminating or repeating). Consider this intriguing thought: Are there decimal expansions that *cannot* be converted into an equivalent fraction? We can rephrase this question: Are there numbers that are *not* rational?

Let's consider this decimal expansion:

0.1010010001000010000010000001 ...

This expansion has a rule. What is it? There is one more 0 after each successive 1. With this rule, anyone could continue writing the decimal expansion. Although this expansion has a rule, *we cannot find any recurring groups in it*. In other words, we cannot find any *repeating patterns*. This number, therefore, *cannot be the expansion of a fraction*. Its partial sums,

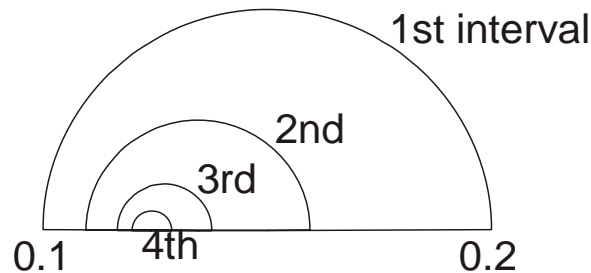
$$\frac{1}{10} + \frac{1}{1,000} + \frac{1}{1,000,000} + \frac{1}{10,000,000,000} + \dots$$

do *not* converge to any rational number.

Although the partial sums do not converge to any rational number, we can show that this number does converge to some sort of *gap* in the set of rational numbers. You might be saying to yourself, "Gap? A gap in the set of rational numbers? I thought these numbers were everywhere dense. How can there be a gap (or gaps) in them?"

Consider our number. First, we know that this number is less than 0.2 (or less than 0.19999...). As we consider the other partial sums, we know that they will all lie between 0.1 and 0.2 on the number line. Using the same rationale, we know that this number (in the thousandths position) will lie between 0.101 and 0.102. We are narrowing down our culprit. Consider the next appearance of 1 in the expansion. From this we know that our number lies between 0.101001 and 0.101002. Next, we know that our number lies between 0.1010010001 and 0.1010010002. These intervals get small very quickly. It looks like this:

Note: This essay is extracted from a Lesson from the forthcoming textbook *Mathematics: Building on Foundations*.



The lengths of our intervals can be tabulated as follows:

<i>Interval</i>	<i>Length</i>
1 <sup>st</sup>	0.1
2 <sup>nd</sup>	0.001

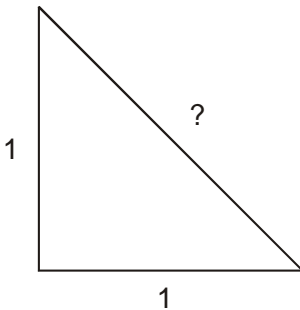
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Interval	Length
3 <sup>rd</sup>	0.000001
4 <sup>th</sup>	0.0000000001

These lengths *converge* to zero. Our larger partial sums will crowd into these ever-decreasing intervals that become smaller and smaller at “battle speed.” Our number is located at the converging point. We have constructed a number that does not converge to a rational number, yet this number *converges to a point on the number line* (which is everywhere dense with rational numbers!). Stop and consider that.

This point, considered at infinity, is a certain determinable distance from zero. Yet, we cannot measure this distance in whole units or in fractions of whole units. We have no way of fitting this number in  $\mathbb{Q}$ , the set of rational numbers. Because of this conundrum, mathematicians have constructed a new set of numbers to handle this situation. The set is appropriately called the set of **irrational** numbers (denoted by the letter **I**). Irrational numbers cannot be written as the ratio of two integers. Their decimal expansions are infinite, without pattern, but given by some rule so that we can say that the expansion represents a point at a definite distance from 0 on the number line. Irrational numbers are non-rational or non-ratio numbers.



The ancient Greeks, namely the students and followers of Pythagoras (ca. 582-ca. 500 BC), first discovered irrational numbers (thanks to some discoveries made by the ancient Egyptians). These numbers made their encore appearance as these mathematicians worked with their beloved right angle triangle. Recall the Pythagorean Theorem: the sum of the squares of the two legs of a right angle triangle equal the square of the hypotenuse. If the legs of a right angle triangle measure  $a$  and  $b$  respectively and the hypotenuse measures  $c$ , then  $c^2 = a^2 + b^2$ . The famous 3-4-5 right-angled triangle fits this formula like

a glove:  $5^2 = 3^2 + 4^2$  or  $25 = 9 + 16$ .

Enter Mr. Spoiler. He asks Mr. Pythagoras, “What is the length of the hypotenuse when the legs are both equal to 1?” Mr. Pythagoras responds, “Obviously, we can construct such a right angle triangle. I take my compass and ruler and mark off and measure 1. I draw the two legs of measure 1. I connect the third edge of the triangle with a straight line. Yes, the hypotenuse is a measurable distance.”

Mr. Spoiler, “Well, Mr. Pythagoras, what is the length of the hypotenuse?” Mr. Pythagoras takes fingers to the sand and starts calculating:

$$\begin{aligned}1^2 + 1^2 &= c^2 \\1 + 1 &= c^2 \\2 &= c^2\end{aligned}$$

Mr. Pythagoras suddenly lets out an anguished cry. He grabs his chest and falls to one knee. Mr. Spoiler cries out, “911! 911!” An hour later a chariot rolls up. “What’s the matter?” asks Dr. Bones. Mr. Spoiler answers, “Mr. Pythagoras seems to be in a state of distress. Can you help him?” Dr. Bones replies, “Let me take a look.” Dr. Bones gives Mr. Pythagoras a thorough physical exam. After comparing what he observes with his scroll of medical terms and definitions, he turns to Mr. Spoiler and says, “It’s serious. We must operate immediately.” “What are you going to do?” queries Mr. Spoiler. “Extract the square root of 2,” answers Dr. Bones. So, Dr. Bones places the limp body of the still conscious Mr. Pythagoras on his chariot and rides off. As they disappear into the sunset, Mr. Spoiler hears a fading, yet heart-rending cry from the lips of Mr. Pythagoras, “But you can’t! You can’t! You can’t extract the square root of 2!”

Pythagoras was committed to a worldview that saw the natural numbers as generating the particular things of the universe. “Numbers rule the universe” or “number is all” was the cry of the brotherhood of

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the Pythagoreans. Pythagoras and his disciples erred in thinking that numbers generate the particular things of the universe. Numbers are a *tool* that can only *report* on the particular things of the universe. For the Pythagoreans, natural numbers and fractions (the ratio of counting numbers) formed the generating foundation of all existence. Given this worldview, it is no surprise that  $\sqrt{2}$  gave them heartburn. No natural number (or ratio of natural numbers) multiplied by itself *exactly* equals 2.  $1.4 \left(1\frac{2}{5}\right)$  is close because  $(1.4)^2 = 1.96$ .

$1.5 \left(1\frac{1}{2}\right)$  is close too because  $(1.5)^2 = 2.25$ . So,

we know that  $\sqrt{2}$  is between 1.4 and 1.5. Does  $\sqrt{2}$  exactly equal a rational number whose decimal expansion either terminates or repeats? No.

... an irrational number ... lies hidden in a kind of cloud of infinity.

Michael Stifel (1487-1567)

This is how we prove that  $\sqrt{2}$  is *not* a rational number. This proof, first devised by the school of Pythagoras, uses a method of proof called, in Latin, *reductio ad absurdum* (literally means “bringing back from absurdity”). This method of reasoning is also called *indirect proof*. To describe this method, we have to engage the notion of the *negation* of a statement. For a given statement S that is provable, its negation is a statement meaning the falsity of S. For example, if S is the statement “x is a negative integer,” then  $\sim S$  (the negation of S), is the statement “x is a positive integer or zero.” Recall that one of the transcendental laws of logic, called the *law of the excluded middle*, asserts that for any given statement that is provable, either the statement itself or its negation is true. Another transcendental law, the *law of contradiction*, asserts that a provable statement and its negation *cannot be true simultaneously*. Therefore, according to these two laws, for every statement S, exactly one of the statements, S or  $\sim S$ , is true. In particular, if  $\sim S$  is not true, then S must be true. These preliminary laws serve as the basis for the indirect proof or proof by *reductio ad absurdum*. With this method you start by assuming the opposite of what you want to prove (i.e., the negation of the conclusion). In this case, we start by assuming that  $\sqrt{2}$  is a rational number. Given this assumption, we proceed to reason, on the basis of the hypothesis, to a point of contradiction. Once we get a contradiction, this shows that the hypothesis and the negation of the conclusion are contradictory. We then recognize that our assumption is false (i.e., the negation of the conclusion) and therefore, by the law of contraposition, the conclusion must be true. Hence, we have proved what we have intended to prove.

The fictional detective Sherlock Holmes employed reasoning similar to the *reductio ad absurdum* method in solving crimes. He stated, “When you have eliminated the impossible, whatever remains, however improbable, must be the truth.” In the context of Holmesian wisdom, *reductio ad absurdum* means that we eliminate an assumption by reasoning from this assumption to a contradiction. Once we have done this, the assumption is eliminated and we must accept the negation of the assumption as true.

In the annals of mathematical proof, there are three ways to prove that  $\sqrt{2}$  is irrational. The method popular in modern texts follows an algebraic line of reasoning. The original proof, thought to be used by the followers of Pythagoras, was based upon geometric arguments.

First, we shall assume that  $\sqrt{2}$  is a rational number. We must now reason to a contradiction.

Second, if  $\sqrt{2}$  is a rational number, then, by definition,  $\sqrt{2}$  can be written as a ratio of two distinct integers. We get:

Equation 1.  $\sqrt{2} = \frac{a}{b}$  where  $a, b \in \mathbb{Z}$  and  $b \neq 0$ .

# Proof that $\sqrt{2}$ is irrational

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We will also assert that this ratio,  $\frac{a}{b}$ , is a fraction written in lowest terms (it has no common factors). For example, the fraction  $\frac{6}{8}$  is *not* in lowest terms because the numerator and denominator have common factors other than 1, namely 2.  $\frac{3}{4}$  is in lowest terms because it has no common factors other than 1; the greatest common factor (GCF) of 3 and 4 is 1. Using number theory terminology, 3 and 4 are relatively prime. We require that  $a$  and  $b$  are relatively prime; i.e., the GCF of  $a$  and  $b$  is 1.

Third, from Equation 1, we square both sides of the equation. We get:

$$\text{Equation 2. } 2 = \frac{a^2}{b^2}$$

Fourth, from Equation 2, we multiply each side of the equation by  $b^2$ . We get:

$$\text{Equation 3. } 2b^2 = a^2$$

What conclusions can we draw from Equation 3? Since  $b$  is an integer, then  $b^2$  must be an integer and  $2b^2$  must be an *even* integer. Why? Even integers are multiples of 2. Since  $2b^2 = a^2$ , then we can conclude that  $a^2$  must be an *even* integer.

Fifth, since we have established that  $a^2$  is an even integer, then what can we say about  $a$ ?  $a$  must be also an even integer. Why? If we take an odd integer and square it, will we ever get an even integer? *Never*. Hence,  $a$  must be an even integer. If  $a$  is an even number, then  $b$  must be an odd number since we agreed that  $a$  and  $b$  had no common factors (other than 1). To reiterate,  $b$  must be an *odd integer*.

Sixth, since  $a$  is an even number it can be written as a multiple of 2. We shall let  $a = 2r$  where  $r \in \mathbb{Z}$ .

Seventh, we substitute  $2r$  for  $a$  in Equation 3. We get:

$$\text{Equation 4. } 2b^2 = (2r)^2 \text{ or } 2b^2 = 4r^2$$

Eighth, from Equation 4, we divide both sides by 2. We get:

$$\text{Equation 5. } b^2 = 2r^2$$

What conclusions can we draw from Equation 5? By our previous reasoning, that  $b^2$  must be an even integer.

Ninth, since  $b^2$  is an even number, then  $b$  must be an even integer. The dénouement of the drama is now upon us. Stating that  $b$  must be an even integer contradicts what we said earlier that  $b$  must be an *odd integer*.

Therefore, when we assumed that  $\sqrt{2}$  is a rational number, we reasoned to a contradiction. This tells us that  $\sqrt{2}$  is a *not* a rational number. QED!<sup>1</sup> This logic behind this proof is classic. The proof that  $\sqrt{2}$  is irrational is considered to be a member of a special class of mathematical proofs; proofs that are beautiful because they are elegant.

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<sup>1</sup> Sometimes a QED can be lethal. According to legend, Hippasus, a follower of Pythagoras, was thrown overboard and drowned at sea for unveiling a number that is *not* equal to a ratio of two numbers.

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The Greeks called  $\sqrt{2}$  an *incommensurable* length because it could not be represented as a ratio of two integers. In our construction of a right angle triangle with legs of measure 1, the hypotenuse definitely has a measurable length. We can park this “number horse of a different color” on a point on the number line just like we could pinpoint 0.1010010001000010000010000001... on a point on the number line.

