Mathematics: The Utility of Beauty

by James D. Nickel

> Now we've come to something really spooky. The applied mathematics process works ... Time and again, the applied mathematics process demonstrates the unreasonable effectiveness of mathematics in the natural sciences ... How can this be? The miracle itself is wonder enough. Wigner wrote: "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve." Yes. But if, as I claim, the motivation for the development of mathematics is primarily aesthetic, and not utilitarian, then the wonder compounds itself. We are talking about the paradox of the utility of beauty. And we are dealing with a miracle of second order of magnitude.
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> Jerry P. King, *The Art of Mathematics*, p. 121.

In Professor Jerry P. King's book, *The Art of Mathematics* (New York: Dover Publications, [1993] 2006), he establishes, in a persuasive 300-page argument, the existence and nature of mathematical beauty. Mathematicians have long recognized *both* the utility and the beauty of the manifold patterns revealed in the analysis of number and space, but their attempt to leap from recognition to *explanation* has resulted in many startling remarks. One of the most remarkable attempts at "explanation" was made by the eminent scientist Albert Einstein (1879-1955):

> "The eternal mystery of the world is its comprehensibility."[1]

To Einstein, the fact that we can comprehend the world was not only a mystery; *it is an eternal mystery*. Professor King, in his words quoted above, notes the miracle of utility; i.e., the workability of applied mathematics. To him, the connection between pure mathematics and the physical world is one of miraculous wonder, a wonder of first magnitude. Then King *compounds* this first magnitude wonder by noting that the nature mathematics is not only eminently utilitarian; *it is astonishingly beautiful*. To King, both the utility and beauty of mathematics is a "spooky" puzzle that cannot be solved. He can only affirm the paradox of the *utility of beauty*, an even greater wonder, a *second* order of magnitude wonder.

For the Biblical Christian, the paradox of the utility of beauty is resolved in the nature of creation and the nature of the Creator. Psalm 105 speaks of creation as God's *wondrous* works. According to Colossians 1, the created realm, existing in, through, and by Christ, includes things visible and things invisible. God created all things through the Word (the *logos*[2]); that is, Christ (Genesis 1; John 1). Christ, the eminently rational Word through whom all things of the world were made, is the *mediator* of creation. In Christ, all things hold together. In Christ, all things consist. In Christ, all things have meaning. In Christ, all things are *ultimately rational and comprehensible*.

Hence, for the Biblical Christian, one can expect to find connections, unity in diversity, meaning, applicability, and wonder. We can expect unity between what goes on in our heads (mathematical propositions) and what goes on outside of our heads (the ordered patterns of the physical creation that can be modeled using mathematics). We can expect that mathematics is applicable as a tool to both describe and predict the workings of God's universe. *We can expect utility*.

Not only can we expect utility, *we can also expect beauty*. According to Psalm 27:4, there is beauty in the Lord God, the creator, sustainer, and redeemer. This beauty is reflected in the wonder of His created realm. In spite of the fallen nature of this realm, we marvel at its magnificence as we behold a new-born baby, a meadow replete with flowers, the precision of the honeycomb, a gentle snow-fall draping our world with

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[1] Albert Einstein, *Out of My Later Years* (New York: Citadel Press, [1950, 1956, 1984] 1991), p. 61.
[2] The "first order of magnitude" definition of *logos* is "eminent rationality."

pure white, the radiant colors of the rainbow, the majestic rising of a harvest moon, the deafening power of a waterfall, and the resplendence of a sunrise and sunset. There is poetic resonance in creation. The author of such beauty is the Creator God, an extravagantly good God. His generosity revealed in the created order is His goodness raised to the power of infinity.

Beauty is revealed in God's handiwork (Psalm 19:1), things seen and also things *unseen*. The world of mathematics is primarily an invisible world, a world consisting of numerical and spatial patterns, a world primarily of rational and logical (*logos*) thought. This rational and logical world, with its invisible patterns and connections, also *holds together in the Logos of God in Christ*. Mathematicians have long known the beauty of mathematical objects although they rarely been able truly justify the reason for this beauty. Anyone who has taken sufficient time to "smell the flowers" of the classical proofs of mathematics …

the many proofs of the Pythagorean Theorem …

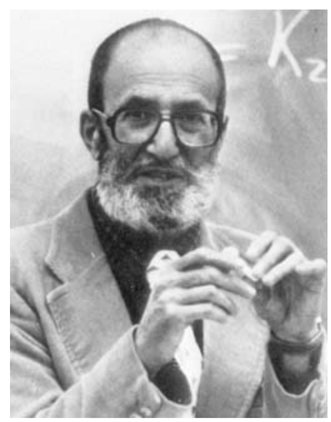
the proof that $\sqrt{2}$ is an irrational number …

the proof the number of prime numbers is infinite …

will smell the aroma of *elegance* and *poetry*.

Professor King takes considerable space in his book to define and illustrate the meaning of mathematical elegance. To him, a mathematical object (a proof, a relationship, an equation, etc.) is elegant if it reveals (1) the principle of minimal completeness and (2) the principle of maximal applicability.[3]

To illustrate elegance, King exegetes one of the remarkable observations made by the Hungarian-born American mathematician Paul Halmos (1916-2006). Consider a single-elimination[4] basketball tournament that begins with four teams. How many games need to be played to determine a champion? With four teams, it is easy. In the first round, two games are played and two teams, the losers, are eliminated. In the second round, the remaining two teams play to determine the champion. In total, three games (2 + 1 = 3) are played to determine the winner. For eight teams, four games are played in the first round, two in the second round, and one in the final round. In total, 4 + 2 + 1 = 7 games are played. Now consider 3789 teams. How many games must be played to determine the champion in single-elimination? At this point, our minds swagger with the multitude of steps that need to be taken to calculate the answer. Professor Halmos' analysis is an example of an *elegant* solution. He noticed that each game has exactly one loser and that each team, except the tournament champion, will lose exactly one game. Hence, the total number of games played is equal to the number of losing teams! If we begin with 3789 teams, there will be one champion and 3788 teams have lost. Therefore, 3788 games are played! To quote Professor King:



Paul Halmos, Public Domain

> Halmos says this "pure thought" method of solution is "pretty." An understatement, Mr. Halmos. It's elegant.[5]

Why is this solution elegant? First, it meets the requirement of minimal completeness. The solution to the number of games to be played in any single-elimination tournament can be written using exactly nine words: *The number of games equals the number of losers*. Second, the solution meets the requirement of maximal applicability because the solution depends upon an idea that is *replete* within *every* branch of mathematics, the

[3] King, p. 181.

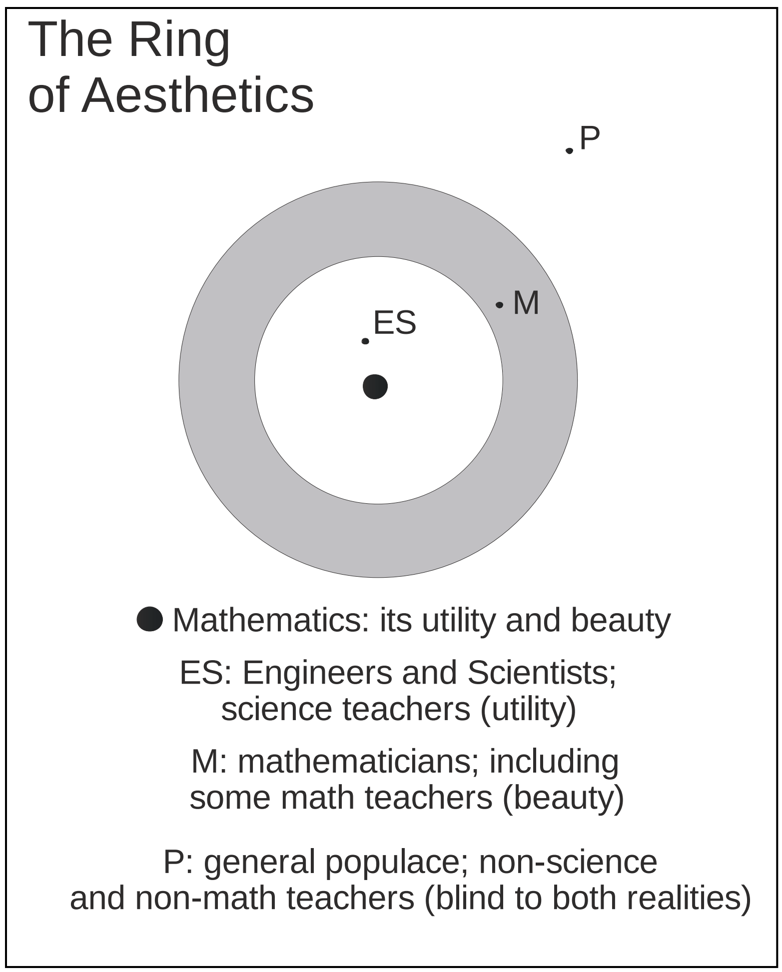
[4] Teams that play in a single-elimination tournament are eliminated at the first lose of a match or game.

[5] King, p. 183.

notion of one-to-one correspondence between two groups of objects. With 3788 teams, counting the number of games *is not easy*. Yet, Halmos's solution is elegant because counting the number of losers is trivial! According to King, "the set of losing teams and the set of games played may be placed in one-to-one correspondence with each other."[6] You use the principle of one-to-one correspondence when you walk into a large auditorium or movie theater and note, just by looking (and not by counting), whether or not the number of people equals the number of chairs. You know that the auditorium is not full by seeing empty chairs! If there are empty chairs and if p = number of people and c = number chairs, then $p < c$. If there are no empty chairs, then $p = c$. If there are people standing and no empty chairs, then $p > c$.

Without going into the detail, the proof that $\sqrt{2}$ is irrational is elegant. The proof that the number of prime numbers is infinite is elegant. The equation $e^{i\pi} + 1 = 0$ is elegant. Isaac Newton's universal law of gravitation, $F = G\frac{m_1 m_2}{d^2}$, is elegant. Because of this elegance, there is *beauty* in mathematics.

The connection between beauty and mathematics is an oxymoron to a majority of people, all of whom have had, in our modern world, some form of mathematical training (from K to University). In his book, Professor King uses two concentric circles to help picture the dilemma that is facing mathematics educators. The large filled dot in the center represents the world of mathematics, its utility and beauty. The annulus, or ring-like structure, represents the tribe of mathematicians who are what they are primarily because mathematics is, to them, a beautiful world.[7] This ring King denotes as the "ring of aesthetics." Mathematicians see beauty in mathematics because they stand "far enough" away from the subject to grasp a portion of its interconnected totality. They have the ability and training to stand back and gaze at its supreme beauty just as an art connoisseur in the Louvre appreciates the Da Vinci's *Mona Lisa*.



The people in the inner circle, a people very close to mathematics because they use it day by day, are the ES types (the engineers and scientists). Most engineers and scientists appreciate mathematics solely and only because of its utility. *Because they are so close to the mathematical world, they often fail to appreciate its beauty*. For them, mathematics is all about algorithms, routines, and calculating answers. John Saxon (1923-1996), the

[6] King, p. 184.

[7] Unfortunately, not all math teachers understand or are able to teach the beauty of the mathematical world and/or the utility of the mathematical world.

originator of the popular Saxon mathematics program, was an *engineer*. Hence, his texts represent a "cook-book" approach to mathematics. They are like Chilton car manuals; they teach the "nuts and bolts" of computation but they do *not* reveal the beauty of mathematics. More than one college mathematics professor has told me that the Saxon math program teaches *ugly* mathematics.

The point outside the two concentric circles, point P, represents the general populace (including most non-science and non-math teachers) *who are generally blind both to the utility and beauty of mathematics*. These "P people" are unfortunately in the majority. They are the people who generally "hate math." These are the people who say, "What's the use of math?" And, if you tell these people that "mathematics is beautiful," they will call for the men in white suits to strap you down and take you away on a stretcher.

Biblical Christian educators are called by God to unfold before their students the nature of God, His ways and purposes, and the nature of His creation (visible and invisible). It is only the Biblical Christian educator who can truly account for King's "paradox of the utility of beauty." It is only Biblical Christian educator who can endeavor to teach the "utility of beauty" in Biblical balance, not absolutizing one aspect over the other, i.e., not teaching utility *only* or beauty *only* but the *utility of beauty*.

Hence, Biblical Christian educators (parents, headmasters, board members, pastors, support staff, and teachers from K to University) have a difficult charge in front on them. They all must recognize the "utility of beauty" nature of mathematics and its importance in the school curricula. Although mathematics is never to be absolutized as an "end all and be all" of the curricula, it must not be minimized or diminished in favor of other subjects or extra-curricular activities. I summarize this brief essay with recommendations for thought and action:

1. From K-6, arithmetic must be carefully taught "line upon line, precept upon precept." Careful attention must be paid to the *beauty* of the fine nuances of arithmetic; i.e., the *beauty of its utility*. According to the Israeli mathematician Ron Aharoni, the author of *Arithmetic for Parents*, a resource that every headmaster, teacher, and parent should study:

   > … elementary mathematics isn't simple at all. It has depth and beauty … Proper teaching of mathematics depends more on an understanding of the mathematical principles than on educational tricks. It requires familiarity with the way the fine mathematical layers lie one upon the other.[8]
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   > The elementary arithmetic we learned as children contains some of the most beautiful mathematical discoveries ever made. Why, then, is it not perceived by most people as beautiful? Mainly because it is often learned mechanically, in a way that does not reveal its beauty.[9]

2. Teachers need to know the *utility* of mathematics.[10] They should be able to connect it to the real world of their student's experiences and to the real world of providential history, covenantal science, the *total truth* of the Biblical Christian world view, and the fascination of mathematical biography. Many math teachers surprisingly do not know much of these applicable real world connections, connections that will resurrect their teaching with life and motivation.

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[8] Ron Aharoni, *Arithmetic for Parents: A Book for Grownups about Children's Mathematics* (El Cerrito, CA: Sumizdat, 2007), pp. v-vi.
[9] *Ibid*., p. 16.
[10] *Mathematics and the Physical World,* written by Morris Kline in 1959 (reprinted by Dover Publications), is a good place to start if you need to learn this utility.

3. Teachers need to know the *beauty* of mathematics.[11] For example, when teaching fractions we need to teach their applicable utility (primarily in measurement). But there is beauty hidden in those numerators and denominators. Consider this amazing equation:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots=1$$

Here we note that an infinite sum of fractions that follow a certain pattern sum to a finite number and that number is 1! This is wonder! This is beauty! The proof that this equation is true is truly *elegant*. Here it is:

$$\frac{1}{1\cdot 2}=\frac{1}{1}-\frac{1}{2} \text{ because } \frac{1}{1}-\frac{1}{2}=\frac{1\cdot 2-1\cdot 1}{1\cdot 2}=\frac{2-1}{2}=\frac{1}{2}$$

Likewise, $\frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3}$ because $\frac{1}{2}-\frac{1}{3}=\frac{1\cdot 3-1\cdot 2}{2\cdot 3}=\frac{3-2}{6}=\frac{1}{6}$

and $\frac{1}{3\cdot 4}=\frac{1}{3}-\frac{1}{4}$ because $\frac{1}{3}-\frac{1}{4}=\frac{1\cdot 4-1\cdot 3}{3\cdot 4}=\frac{4-3}{12}=\frac{1}{12}$

We want to find the sum of $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots$.

By substitution, we get: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots$

We note that $-\frac{1}{2}+\frac{1}{2}=0$, $-\frac{1}{3}+\frac{1}{3}=0$, $-\frac{1}{4}+\frac{1}{4}=0$, etc., *ad infinitum.*

With this in mind, we note: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots=\frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\ldots$

Hence, $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots=\frac{1}{1}+0=1$

This proof can be understood by any child who knows how to add or subtract proper fractions and proofs like this need to be part of our instruction in the "utility of beauty."

4. Non-math and non-science teachers need to appreciate this mathematical "utility of beauty" paradigm because the same wonder is revealed in the study of grammar, history, language, logic, theology, rhetoric, etc.

[11] The textbooks written by Harold R. Jacobs (*Elementary Algebra, Geometry: Seeing, Doing, Understanding, and Mathematics: A Human Endeavor*) employ the "beauty of utility" theme magnificently.