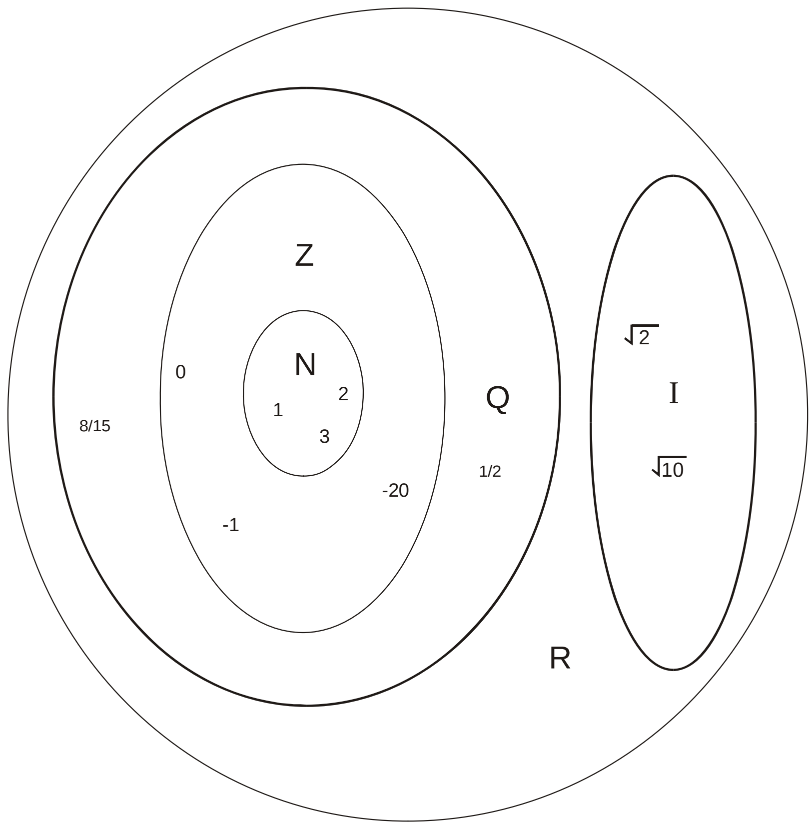
THE POWER OF THE CONTINUUM

BY JAMES D. NICKEL

Irrational numbers (e.g., $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and π) appear as gaps, indeed many gaps, in our number line of *everywhere dense rational numbers*. This makes sense because two fractions must always be different or else we would have no way of telling them apart. No matter how close together we pack the fractions, *there must always be some tiny gap between them*. The set of irrational numbers effectively fills all these gaps so that we can say, "The line is truly filled up." Mathematicians call the union (symbol ∪) of the set of irrational number with the set of rational numbers the set of *real numbers* (denoted by ℝ). Using symbols, I ∪ ℚ = ℝ. Note that ℚ is *not* a subset of I. All the elements in ℚ are distinct from the elements in I. ℚ is *not* contained in I. In symbols, ℚ ⊄ I.

Note: This essay is extracted from a Lesson from the forthcoming textbook *Mathematics: Building on Foundations*.

Mathematicians use the word *continuum* to denote the set of real numbers, the union of the set of rational numbers with the set of irrational numbers.[1] This word makes sense because the set of real numbers "fills the number line" completely (no gaps are left). Note these remarks about the continuum made by the mathematician Tobias Dantzig:



> Indeed, whether we use a ruler or a weighing balance, a pressure gauge or a thermometer, a compass or a voltmeter, we are always measuring what appears to us to be a *continuum*, and we are measuring it by means of a graduated *number scale*. We are then assuming that there exists a perfect correspondence between the possible states within this continuum and the aggregate [set – J.N.] of numbers at our disposal; … Therefore, any measuring device, however simple and natural it may appear to us, implies the whole apparatus of the arithmetic of real numbers: behind any scientific instrument there is the master-instrument, arithmetic, without which the special device can neither be used nor even conceived.[2]

Based upon Dantzig's comments, we see that the real numbers, or the continuum, *report* on the way God has made the universe. We construct measuring devices that signify a quantitative state and, in principle, these measuring devices, no matter how simple, assume the continuum as their foundation.

[1] Continuum comes from a Latin word meaning "continuous."

[2] Tobias Dantzig, *Number: The Language of Science* (Garden City: Doubleday Anchor Books, [1930] 1954), pp. 245-246.

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The real number continuum introduces another important mathematical property called *closure*. Closure is the property that a set of numbers reflects when that set is closed with respect to a particular arithmetic operation. For example, the set of natural numbers is closed with respect to the operation of addition. This means that the sum of any two natural numbers is always a natural number. A little more reflection will lead you to conclude that the set of natural numbers is also closed with respect to multiplication. How would you justify this? What about the operation of subtraction? The set of natural numbers is *not* closed with respect to the operation of subtraction (e.g., 3 – 5 is *not* a natural number). We extend our number system to include integers and the set of integers is closed under the operation of subtraction. What about the operation of division? We should easily conclude that the set of natural numbers is *not* closed under division (e.g., 8 ÷ 5 is *not* a natural number). Is the set of integers closed under division? Because the set of integers is *not* closed under division, we extend the set of integers to include the set of rational numbers. The set of real numbers (the union of the set of irrational numbers and the set of rational numbers) is closed under the four arithmetical operations: (1) addition, (2) subtraction, (3) multiplication, and (4) division. In symbols, we state these closure properties as follows:

1. If $a, b \in \mathbb{R}$, then $a + b \in \mathbb{R}$.
2. If $a, b \in \mathbb{R}$, then $a - b \in \mathbb{R}$.
3. If $a, b \in \mathbb{R}$, then $ab \in \mathbb{R}$.
4. If $a, b \in \mathbb{R}$, then $\frac{a}{b} \in \mathbb{R}$ where $b \neq 0$.

In algebraic operation we use *properties of equality* continuously; in fact, almost subconsciously. These properties are based upon the "balance beam" principle; i.e., the operation you perform on one member of an equation must also be performed on the other member in order to keep the equation true or "balanced." Let's formally define these properties.

1. *Reflexive* property. This property is derived from the meaning of the equal sign (=). Whenever we see this sign, it should tell us that the quantity on the left side (left member) is equal to (or reflects) the quantity on the right side (right member) of the equation. For example, $y = x^2$ means that the quantity represented by *y* reflects (or is equal to) the quantity represented by x^2. When you solve an equation for x, you are incorporating the reflexive property unconsciously. For example, the reflexive property guarantees that you can solve an equation like $x + 5x + 6 = 18$ for x. In symbols, the reflexive property of equality states that $a = a$. Pretty simple, isn't it?
2. *Symmetric* property. This property allows you to exchange the two members of an equation. For example, $4x - 7 = 9 - 7x + 15$ can be exchanged as follows: $9 - 7x + 15 = 4x - 7$. In symbols, the symmetric property of equality states that if $a = b$, then $b = a$. Again, this property is intuitively obvious when you understand the principle of equality (the balance beam).
3. *Transitive* property (similar to *syllogistic* reasoning). This property was noted centuries ago by Euclid. In his first axiom, he stated, "Things being equal to the same thing are also equal to each other." We also used this property in algebra. For example, given the two equations $5x - 6 = y$ and $y = 3x$, we can eliminate the common term *y* and connect the two equations into one; i.e., $5x - 6 = 3x$. We called this method solving equations by *substitution*, but we are really using Euclid's ancient axiom. If symbols, the transitive property of equality states that if $a = b$ and $b = c$, then $a = c$.
4. *Addition* property. This property was also noted by Euclid. In his second axiom, he stated, "If equals are added to equals, the wholes are equal." Again, we use this property many times in algebra. It justifies the addition (or subtraction) of any number or algebraic term to any equation *as long as you add*

(or subtract) it to both members to keep the equation in balance. For example, given the equation 5x + 4 = 14. To solve for x, you can subtract 4 from both sides of the equation and get 5x = 10. In symbols, the addition property of equality states if a = b, then a + c = b + c.

5. *Multiplication* property. This property is similar to the addition property. It justifies the multiplication of both sides of an equation by a non-zero value. For example, given the equation 5x = 10, we divide both sides of the equation by 5 (or multiply by $\frac{1}{5}$) to solve for x; i.e., x = 2. In symbols, the multiplication property of equality states that if a = b, then ac = bc when $c \neq 0$.
6. *Multiplication property of zero*. This property states that the product of any number and zero is zero. In symbols, a(0) = 0 and (0)a = 0.
7. *Double negative* property. In symbols, -(-a) = a.

With the properties of closure and equality under our belt, let us now consider irrational numbers as a whole. How many are there? Can we count them like we can count rational numbers? That is, are they *denumerable* like the set of rational numbers? So far, we have come across irrational numbers haphazardly. There must be a lot of them since the majority of whole numbers are *not* perfect squares and the square roots of these numbers are irrational numbers.

Maybe the number of rational numbers equals the number of irrational numbers; i.e. they have the same cardinality, denoted $\aleph_0$ by the German mathematician Georg Cantor (1845-1918). That fact that Q and I are distinct (Q is not a subset of I) ought to make you wonder about that assumption. Let's return to work of Cantor and see how he resolved the question, "Are the irrational numbers denumerable?"

Cantor's method has been hailed as one of the most elegant and ingenious proofs in the history of mathematics. First, let's consider the set of real numbers and let us imagine them all in their decimal expansions. Of these numbers, let us restrict ourselves to the real numbers (both rational and irrational) between 0 and 1; i.e., to the numbers that begin with 0 in their whole number parts (that means we do not need to bother with integral numbers). Cantor showed that even this section of real numbers is "more numerous" than *all* the natural numbers; i.e., you *cannot* arrange these numbers in a sequence without leaving some real numbers out of it.

To prove this, Cantor used the *reductio ad absurdum* approach. He assumed that you *can* count the real numbers between 0 and 1 and then he reasoned to a contradiction. Follow this reasoning carefully. Your mind needs to be fully engaged. You may need to remove all distractions so that you can focus.

The set of real numbers between 0 and 1 has the form $0.a_1a_2a_3 \ldots$, where the digits after the decimal point may terminate (as in 0.215), repeat infinitely (as in 0.333 …), or be infinite but *not* repeatable (as in $\sqrt{2}$). Note that by this definition we are including all the rational numbers between 0 and 1. Now, let's assume that we can establish a one-to-one correspondence between the real numbers between 0 and 1 and the natural numbers. If we can show that there is a real number that has not been counted by this method, then our assumption is false (the real numbers between 0 and 1 are denumerable) and what we want to prove is true (the real numbers between 0 and 1 are not denumerable).

We now prepare our arrangement of real numbers. Each real number has a *unique* decimal expansion unless the expansion terminates after a finite number of decimal places. In this case, it may be represented by an infinite decimal expansion involving the sequence .999999 … For example, 0.245 = 0.244999999… We must also understand the principle known as the *fundamental property of decimal expansions: If a number has an infinite decimal expansion, no other infinite decimal expansion can represent that number*. This property implies that a *difference in any one digit* in two infinite decimal expansions means that the expansions represent two *distinct* numbers.

Here is our list (notice, we are not writing any numbers; we are establishing a method of operation by this list):

0.	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9 …
0.	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9 …
0.	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9 …
0.	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9 …
0.	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9 …
0.	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9 …
0.	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Cantor then showed that a decimal lying between 0 and 1 *does not appear anywhere on the list*. Let's try to understand the logic he used.

We first consider the number z and the construction of the decimal expansion of z, namely $0.z_1z_2z_3z_4\ldots$ Compare z, with our first row; i.e., $0.a_1a_2a_3a_4\ldots$ We now create a method by which we can determine what digit to put in tenths position of z; i.e., z_1. Our method: if a_1 is 0, 1, 2, 3, 4, 5, 6, or 7, then $z_1 = 8$. If a_1 is 8 or 9, then $z_1 = 1$. z, is now *different* from the number represented in the first row.

Now consider the second row; i.e., $0.b_1b_2b_3b_4\ldots$ We use the same method to determine what digit to put in the hundredths position of z; i.e., z_2. If b_2 is 0, 1, 2, 3, 4, 5, 6, or 7, then $z_2 = 8$. If b_2 is 8 or 9, then $z_2 = 1$. z, is now *different* from the number represented in the second row.

Now consider the third row; i.e., $0.c_1c_2c_3c_4\ldots$ Again, by the same method, we determine what digit to put in the thousandths position of z; i.e., z_3. If c_3 is 0, 1, 2, 3, 4, 5, 6, or 7, then $z_3 = 8$. If c_3 is 8 or 9, then $z_3 = 1$. z, is now *different* from the number represented in the third row.

We continue this reasoning *ad infinitum*. The number we have constructed, namely z, lies between 0 and 1, but is it contained in our enumerated list? It cannot be equal to the number in the first row since, by construction, z_1 differs from a_1. It cannot be equal to the number in the second row since, by construction, z_2 differs from b_2. It cannot be equal to the number in the third row since, by construction, z_3 differs from c_3. And, z cannot be equal to the number in the n^{th} row since, by construction, z_n differs from n^{th} digit of the decimal in the n^{th} place in the enumeration.

By this marvelous piece of logic, Cantor constructed a number *not* in our list. He therefore reached a contradiction and concluded that the real numbers between 0 and 1 are *not* denumerable. If anyone were to try to match the real numbers between 0 and 1 with the natural numbers 1, 2, 3, 4, 5, … by writing them in a sequence, *a real number would always be left out*. In this sense we can say that the real numbers are *more numerous* than the natural numbers. Cantor denoted the number of real numbers by the letter *c*, a letter representing the *power of the continuum*.

Our proof establishes the non-denumerability of both the rational numbers and irrational numbers taken together; i.e., the set of real numbers $\mathbb{R}$. We already know that the rational numbers are countable (they can be written out in the form of a countable sequence).[3] Hence, by implication, the irrational numbers must *not* be countable since if these numbers can be written out in the form of a sequence then we can unite the two sequences by taking numbers alternatively from each to make a new countable sequence. To illustrate this, we can unite the set of positive integers {1, 2, 3, 4,…} with the set of negative integers {-1, -2, -3, -4, …} producing a countable set as follows:

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[3] See the essay *Fractions A Plenty* on www.biblicalchristianworldview.net/Mathematical-Circles/fractionsAPlenty.pdf

1, -1, 2, -2, 3, -3, 4, -4, …

If we could construct a countable sequence consisting of union of the set of rational numbers and the set of irrational numbers, then we could be able to count the set of real numbers. Cantor proved that you cannot do this. Therefore the set of irrational numbers cannot be counted; this set is not denumerable. There are more irrational numbers than rational numbers on the number line.

The irrational numbers spread continuously over the entire number line *in spite of the fact that the rational numbers are everywhere dense!* The set of natural numbers, when compared with the set of rational numbers, *appear* (on the surface) to be as a *few* insignificant needles in a haystack infinitely full of straw (since $\mathbb{N} \subset \mathbb{Q}$). Yet, we discovered that the number of needles *equaled* the number of straws of hay; i.e., a one-to-one correspondence exists between these two infinite sets. In other words, the set of natural numbers and the set of rational numbers are equally denumerable. From Cantor's proof explicated in this essay (i.e., the non-denumerability of the set of irrational numbers), we can now *truly* picture the set of rational numbers as only a *few* insignificant needles in an infinite haystack of irrational numbers. Ponder that!