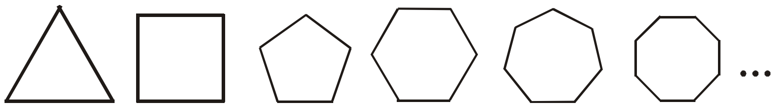
The Platonic Solids: Why Only Five?

by James D. Nickel

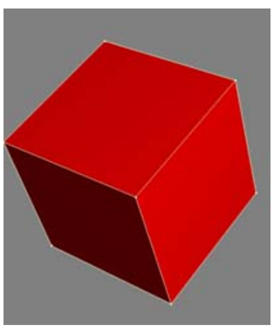
There are technically an *infinite* number of regular convex[1] plane figures (all angles equal and all sides equal) or regular polygons. Below are the first six: equilateral (equal sides) triangle, square, regular pentagon (five angles), regular hexagon (six angles), regular heptagon (seven angles), and regular octagon (eight angles).[2]



All corresponding parts of each of these regular convex figures are equal (i.e., all edges [or sides] and all angles in each figure are equal). Note also that *only two edges meet* at each vertex of each figure.

Let's now shift to the three-dimensional world. Since, in two-dimensions, there are an infinite number of regular polygons, can we, using reasoning by analogy, conclude that, in three-dimensions, there are an infinite number of regular polyhedra, a solid whose faces are all regular polygons? This conclusion about three-dimensions "seems" to be a natural inference from the two-dimensional situation. The way mathematicians think about three dimensional figures like regular polyhedra is to employ the methods of one of its most interesting branches, topology.[3] Using topology, we can study this problem by investigating two unique properties of regular polyhedra: (1) the same number of edges bound each face and (2) the same number of edges meet at every vertex.

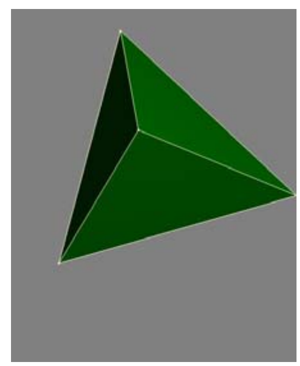
> Note: This essay is extracted from a Lesson from the forthcoming textbook *Mathematics: Building on Foundations*.



Cube (Hexahedron)

To illustrate, picture the cube (a regular polyhedron) at left. The cube has 8 vertices, 6 faces, and 12 edges where 4 edges bound *each* face and 3 edges meet at *each* vertex.

Next, consider the tetrahedron (literally, "four faces" where each face is an equilateral triangle). A tetrahedron has 4 vertices, 4 faces, and 6 edges where 3 edges bound *each* face and 3 edges meet at *each* vertex.



Tetrahedron

Note that the two properties, (1) equal edges bounding each face and (2) equal edges meeting at each vertex, *have nothing to do with size or shape*. Here is where topological tools show us that *only five regular polyhedra* can satisfy these *two requirements*.

We have noted two of these figures, the cube and the tetrahedron. The other three are the octahedron (8 faces; octa from the Greek means "eight"), dodecahedron (12 faces; dodeca from the Greek means "2 plus 10" or "twelve"), and icosahedron (20 faces; icosa from the Greek means "twenty").

---

1 Convex is derived from the Latin *convexus* meaning "rounded." In a convex polygon, the line segment connecting any two points inside the polygon will always stay completely inside the polygon.

2 Note that the number of sides determines the measure of each angle. In general, given a regular polygon with n sides, A, the measure of each angle, is determined by the following formula:

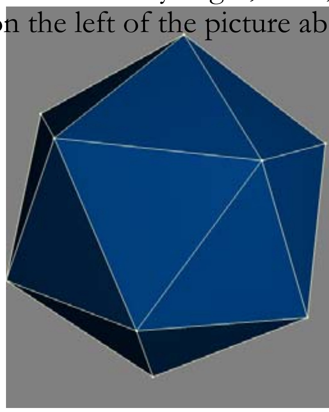
$A = \frac{(n-2)180^o}{n}$. For a square, $A = \frac{(4-2)180^o}{4} = \frac{360^o}{4} = 90^o$.

3 Topology is the study of those properties of geometric forms that remain invariant under certain transformations, as bending or stretching.

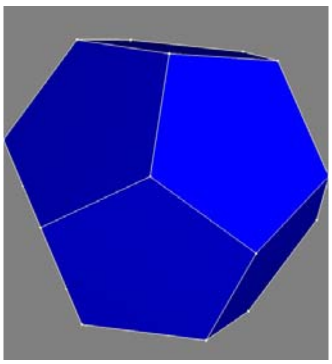
The Platonic Solids: Why Only Five?

by James D. Nickel

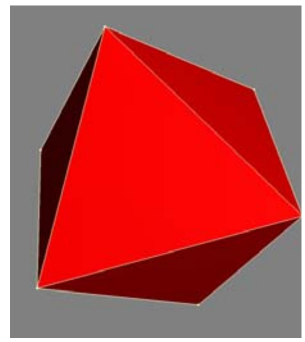
How many edges, faces, and vertices are there in a dodecahedron and an icosahedron, the two figures on the left of the picture above? Instead of just counting (it is easy to get "lost" with these figures), we might want to apply a more systematic approach. For example, we know that the dodecahedron has 12 faces. How many edges on each face? 5. We could calculate the product of 12 and 5 to get the number of edges, but there would be a problem with this approach. How many faces does each edge *share*? 2. Therefore, we've got to divide the product of 12 and 5 by 2 and $\frac{12\times 5}{2}=30$. Similarly, the face of each dodecahedron has five vertices. We could again calculate the product of 12 and 5 to get the number of vertices, but we would encounter the same problem as before; i.e., each vertex shares a face. How many faces does each vertex share? 3. To calculate the number of vertices in a dodecahedron, we compute $\frac{12\times 5}{3}=20$. For the icosahedron, there are 20 faces with each face having 3 edges and each edge shares 2 faces. To calculate the number of edges in an icosahedron, we compute $\frac{20\times 3}{2}=30$. Each of the 20 faces has 3 vertices and each vertex shares 5 faces. To calculate the number of vertices in an icosahedron, we compute $\frac{20\times 3}{5}=12$.



Icosahedron



Dodecahedron



Octahedron

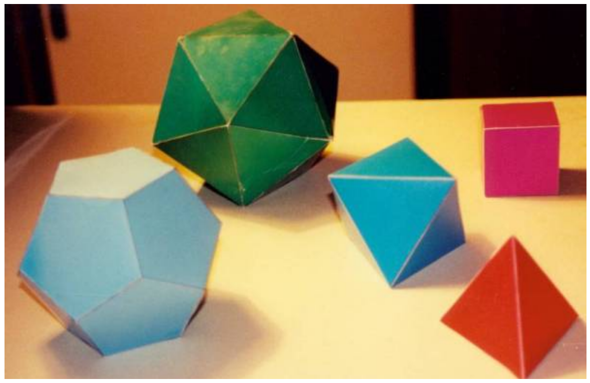
We tabulate our observations as follows (I would highly recommend that these solids be built; there are many sources on the web that will provide templates or "nets" for building them):

Table I

Regular polyhedra	*Shape of face*	*F (number of faces)*	*V (number of vertices)*	*E (number of edges)*
tetrahedron	equilateral triangle	4	4	6
cube	square	6	8	12
octahedron	equilateral triangle	8	6	12
dodecahedron	regular pentagon	12	20	30
icosahedron	equilateral triangle	20	12	30

The Platonic Solids: Why Only Five?

by James D. Nickel



There is a discernible pattern relating the number of faces, vertices, and edges in each regular polyhedron. Can you figure it out? The Swiss mathematician Leonhard Euler (1707-1783), the founder of topology, did. Here is the formula (called the *Euler characteristic*):

$$F + V = E + 2 \text{ or } V - E + F = 2$$

Topology tells us that we cannot reason as we did from an infinite number of regular polygons in two-dimensions to an infinite number of regular polyhedra in three-dimensions. We noted that in two-dimensions, *only two edges meet* at each vertex of each figure. In three-dimensions, you can have any number of edges meet at each vertex and, furthermore, any number of faces. For example, you could have 50 edges meet at a vertex and 6 at another. One face could be a square while another face could be a 40-sided polygon. To restrict a solid figure to equal edges bounding each face and equal edges meeting at each vertex *confines* the number of such figures to *five*.

There are several ways to prove that there are only five Platonic solids.[4] Note, that mathematical proof does not prove anything by exercising rational, independent thought (although this "independence" is assumed by unbelieving mathematicians). Mathematical proof is one way in which man uses logical reasoning, a gift from the Creator God, to justify both visible and invisible patterns of God's law-ordered creation. That there are only five Platonic solids is one of those given patterns.

The proof will employ Euler's characteristic:

Equation 1: $V - E + F = 2$

Let's consider a given polyhedron where s iw the number of sides in each of the faces. It will have F identical faces. Based upon our table, the value of F can be 4, 6, 8, 12, or 20. Counting the total number of edges, we get:

Equation 2. $sF = 2E$

Why? Since each edge belongs to two faces, it is therefore connected twice to sF. We then solve for F. We get:

Equation 3. $F = \dfrac{2E}{s}$

Let's let r = the number of edges that meet at each vertex V. For example, for the octahedron, $V = 6$ and $r = 4$. Counting the total number of edges again, we get:

[4] The Platonic Solids are named after Plato (427-347 BC), the ancient Greek philosopher who was intrigued by the three-dimensional symmetries revealed in these shapes.

Equation 4. $rV = 2E$

Again, since each edge connects two vertices, it is therefore connected twice to rV. We then solve for V. We get:

Equation 5. $V = \dfrac{2E}{r}$

Substituting Equations 3 and 5 into Equation 1, we get:

Equation 6. $\dfrac{2E}{r} - E + \dfrac{2E}{s} = 2$

Using algebra, we can translate Equation 6 into Equation 9 as follows:

Equation 7. $\dfrac{2}{r} - 1 + \dfrac{2}{s} = \dfrac{2}{E}$ (divide both members of Equation 6 by E)

Equation 8. $\dfrac{1}{r} - \dfrac{1}{2} + \dfrac{1}{s} = \dfrac{1}{E}$ (divide both members of Equation 7 by 2)

Equation 9. $\dfrac{1}{r} + \dfrac{1}{s} = \dfrac{1}{E} + \dfrac{1}{2}$ (add $\dfrac{1}{2}$ to members of Equation 8)

Since a polygon must have at least three sides and since at least three edges must meet at each vertex of a polyhedron, then $s \geq 3$ and $r \geq 3$.

Study Equation 9. What is it revealing about r and s? If s and r are *both* greater than 3, we have a contradiction. For example, let $s = 4$ and $r = 4$. Then, we have $\dfrac{1}{4} + \dfrac{1}{4} = \dfrac{1}{E} + \dfrac{1}{2} \Leftrightarrow \dfrac{1}{2} = \dfrac{1}{E} + \dfrac{1}{2} \Leftrightarrow 0 = \dfrac{1}{E} \Leftrightarrow E =$ 0, a contradiction based upon the given parameters of the figures (to make sense, E must be greater than 0). If $s = 5$ and $r = 4$, then $\dfrac{1}{5} + \dfrac{1}{4} = \dfrac{1}{E} + \dfrac{1}{2} \Leftrightarrow \dfrac{9}{20} = \dfrac{1}{E} + \dfrac{1}{2} \Leftrightarrow -\dfrac{1}{20} = \dfrac{1}{E} \Rightarrow E = -20$, another contradiction based upon the given parameters of the figures ($E > 0$). By this reasoning, we only have to find the possible values of r when $s = 3$ and s when $r = 3$.

Setting $s = 3$ in Equation 9, we get:

$$\frac{1}{r} + \frac{1}{3} = \frac{1}{E} + \frac{1}{2} \Leftrightarrow \frac{1}{r} = \frac{1}{E} + \frac{1}{6} \Leftrightarrow \frac{1}{E} = \frac{1}{r} - \frac{1}{6}$$

This equation, $\dfrac{1}{E} = \dfrac{1}{r} - \dfrac{1}{6}$, is true only if $r = 3$, 4, or 5. If $r = 6$, then $E = 0$, a contradiction. If $r > 6$, the $E < 0$, a contradiction. Note the corresponding values of E:

Table II		
r (edges)	*E*	*Regular polyhedra*
3	6	tetrahedron
4	12	octahedron
5	30	icosahedron

Note that the values of E in Table II account for three regular polyhedra in Table I.

Now, let's set $r = 3$ in Equation 9. We get:

The Platonic Solids: Why Only Five?

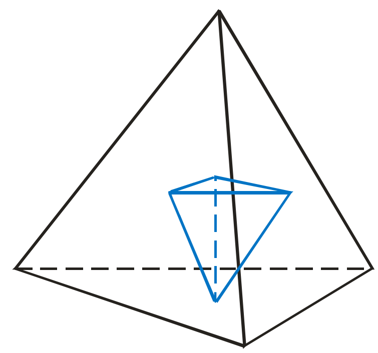
by James D. Nickel

$$\frac{1}{3}+\frac{1}{s}=\frac{1}{E}+\frac{1}{2} \Leftrightarrow \frac{1}{s}=\frac{1}{E}+\frac{1}{6} \Leftrightarrow \frac{1}{E}=\frac{1}{s}-\frac{1}{6}$$

We get the same result in terms of s. This is so because Equation 9 is *symmetric* with regards to the variables s and r meaning these variables can be swapped. Hence, we generate a similar table:

Table III		
s (sides)	*E*	*Regular polyhedra*
3	6	tetrahedron
4	12	cube
5	30	dodecahedron

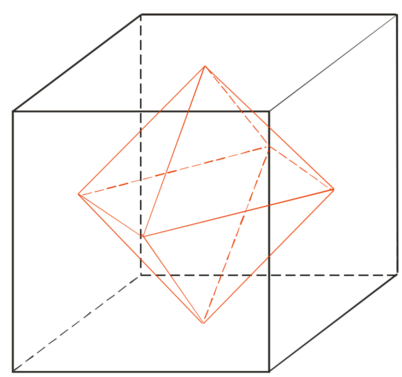
Note also, because of this symmetry, these solids are the duals of the ones obtained when $s = 3$. Duality is a geometrical principle that shows the symmetry between two figures when their parts are interchanged. By the duality principle, for every polyhedron, there exists another polyhedron in which faces and polyhedron vertices occupy complementary locations. This polyhedron is known as the dual, or reciprocal. The tetrahedron is the dual of itself, the octahedron is the dual of the cube (and vice versa), and the icosahedron is the dual of the dodecahedron (and vice versa).



Tetrahedron, dual of itself

Table IV			
r or s	*E*	*Regular polyhedra (r)*	*Dual (s)*
3	6	tetrahedron	tetrahedron
4	12	octahedron	cube
5	30	icosahedron	dodecahedron

These cases exhaust all of the possibilities. Thus, although there are infinitely many regular polygons in a two-dimensional plane, there are only five regular solids in three-dimensional space. QED.



Octahedron, dual of the Cube

Amethyst Crystal (iStockPhoto)

What is even more amazing about the reality that there are only five regular solids in space is that many small details of God's creation reflect this restriction; i.e., it is a creational covenant. Most crystals grow in the beautiful shapes of regular polyhedra. For example, sodium chlorate crystals appear in the shape of cubes and tetrahedra, while chrome alum crystals are in the form of octahedra. Equally fascinating are the appearance of decahedra and idosahedra crystals in the skeletal structures of radiolaria (microscopic sea animals).

Minerals, created by God as a gift to us, are technically defined as homogeneous[5] inorganic[6] solid substances having a definite chemical composition and characteristic crystalline structure, color, and hardness.

[5] Homogeneous, from the Greek, means "same kind or same nature."

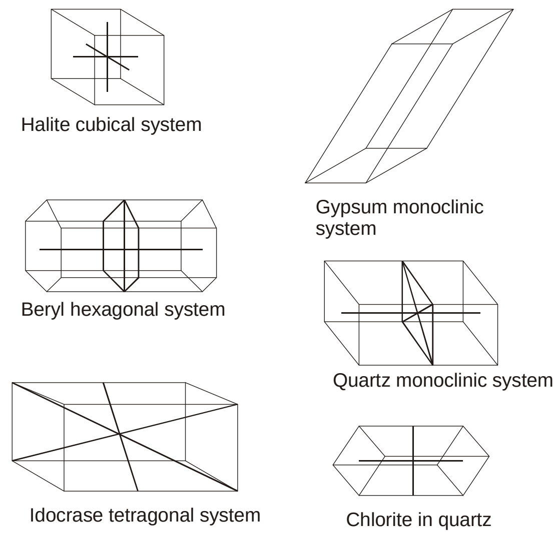
[6] Inorganic means "non-living" as opposed to organic which means "living."

The Platonic Solids: Why Only Five?

by James D. Nickel

Mineralogists study crystals, not only for their intrinsic beauty, but because these structures provide clues to the arrangement of atoms within a mineral thus offering an important means of identification. Only a few minerals, such as opal and silica glass, lack a crystal structure.

Scientists have discovered that all crystals can be placed in one of seven categories called systems. These seven systems are classified as (1) Cubic system, (2) Hexagonal system, (3) Rhombohedral[7] system, (4) Tetragonal[8] system, (5) Orthorhombic[9] system, (6) Monoclinc[10] system, and (7) Triclinic[11] system. These seven systems can be further subdivided into thirty-two symmetry[12] classes and 230 space-groups based on internal structure.[13] Treatment of these subdivisions constitutes the study of crystallography.



Note the observations made by mathematician James R. Newman (1907-1966):

> In the development of crystallography the mathematics of group theory and of symmetry has ... played a remarkable part ... mathematicians by an exhaustive logical analysis of certain properties of space and of the possible transformations (motions) within space [i.e., topology – JN], *decreed the permissible variations of internal structure of crystals before observers were able to discover their actual structure* [italics added – JN]. Mathematics, in other words, not only enunciated the applicable physical laws, but provided an invaluable syllabus or research to guide future experimenters. The history of the physical sciences contains many similar instances of mathematical prevision.[14]

Carefully reread these observations. Let's unpack them from a Biblical viewpoint. Note first that scientists (crystallographers) discovered what mathematicians, by logical analysis, a preveniently decreed. Before these scientists began their explorations, mathematicians had already proved that the symmetry elements of crystals could only be grouped in 32 different ways, *and no others*. Here we see that mathematicians, by logical analysis alone, blazed a trail for crystallographers to follow. In their research of crystals, these scientists confirmed the conclusions of the mathematicians. This fact reflects upon the remarkable connectivity between man's mind (his ability to reason) and the physical world. The mind of man, with his mathematical capabilities, and the physical world, with its mathematical properties, *cohere because of a common Creator*. Hence, we should not be surprised to find a multitude of instances where mathematical conclusions not only connect with, but also *direct* the physical sciences (just as Newman summarizes in the last two sentences of the above quote).

---

[7] A rhombohedron is a solid bounded by six rhombic planes where a rhombus is an oblique-angled equilateral parallelogram; i.e., a rhombus is any equilateral parallelogram except a square.
[8] A tetragon is a polygon having four angles or four sides.
[9] Ortho means "straight" or "right angled."
[10] Monoclinic literally means "one lean."
[11] Triclinic literally means "three leans."
[12] We have already used symmetry two times in this article (algebraic symmetry and topological duality). Symmetry, from the Greek, means "like measure." For a figure to reflect symmetry means that you can determine an exact correspondence of form and constituent configuration on opposite sides of a dividing line or plane or about a center or an axis. Note that butterfly wings are symmetric (the axis of symmetry is the body of the butterfly).
[13] See Philippe Le Corbeiller, "Crystals and the Future of Physics," in *Scientific American* (January 1953), 50-55.
[14] James R. Newman, *The World of Mathematics* (New York: Simon and Schuster, 1956), 2:852-853.