Dominoes and Counting

by James D. Nickel

All of us have an intuitive feeling or innate sense for the counting or natural numbers, including a "sense" for infinity: $\mathbb{N} = \{1, 2, 3, \ldots\}$. The ability to count using numbers is a gift from God. Hence, I like to denote the *sixth* sense as the "sense of counting."

Professional mathematicians have a natural bent for developing structural systems. Hence, they love to *formalize* topics in mathematics. It was not until recently that mathematicians formulated a system of statements that completely defined the set of natural numbers. The reason it took so long is because mathematicians did not feel a need for such a system. Why? Mathematicians did not feel a need for such a system because of the universal and intuitive sense for numbers.

Note: This essay is extracted from a Lesson from the forthcoming textbook *Mathematics: Building on Foundations*.

Giuseppe Peano (Public Domain)

It was the Italian mathematician Giuseppe Peano (1858-1932), in *Arithmetices principia, nova methodo exposita* (1899), who first put together this system of statements in the form of *axioms*[1] (statements that contain words or terms that are left undefined and simply accepted as foundational to the topic under consideration).

Before we inspect the axioms that Peano used to encompass the system of natural numbers, there are three preliminary axioms (axioms behind the axioms, so to speak[2]) or logical requirements that every system of axioms must aim to meet. These requirements are: (1) *consistency*, (2) *independence*, and (3) *completeness*.

First, a set of axioms must be *consistent* in that they do not contradict each other and that it is impossible to derive from these axioms results that will contradict each other. In other words, the set of axioms must comport both with each other and with all the results derived from this set. It should be impossible to derive a proposition, P, and its negation, ~P, from the given axioms.

Second, each axiom in the set must be *independent* of the other axioms in the set. In other words, an axiom in the set cannot be derived from the other axioms in the set. This stipulation guarantees that the set of axioms are minimal (mathematicians like to "spare themselves of extra and unnecessary work").

Third, a set of axioms must be *complete* in that they are sufficient to generate all possible results concerning the given field of knowledge they seek to encompass. This field of knowledge is the aggregate of all statements concerning the original set of *undefined terms*. In other words, a set of axioms is complete if it is impossible to add an additional consistent and independent axiom without introducing additional *undefined terms*.[3]

With these preliminaries established, here is Peano's system[4]:

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[1] Axiom, in Greek, means "to reckon or think worthy."

[2] It is important to note that Biblical presuppositions account for the ability to know mathematics and to use mathematics. In the 19th century, mathematicians established three criteria governing the mathematical system they are trying to develop and/or understand. Then, on the basis of these two foundations (one building upon the other), mathematicians lay out a set of axioms that form the basis for the application of logical thinking and logical derivations. These axioms are also combined with a set of undefined terms (notions that appear to be intuitively obvious). Mathematicians then use these axioms and undefined terms to justify the many derived propositions of the system under study (whether it be a number system or a geometric system).

[3] Although we will not enter into a detailed discussion of the work of the logician and mathematician Kurt Gödel (1906-1978), he showed, in 1930, that any axiomatic system of mathematics (even the system of counting numbers) can either be consistent *or* complete. They *cannot* be both! One, either consistency or completeness, must be sacrificed to have the other. In short, the realm of infinity (e.g.., the natural numbers are infinite in scope) generates such problems in the development of perfectly logical systems. See James D. Nickel, *Mathematics: Is God Silent?* (Vallecito: Ross House Books, [1990] 2001), pp. 183-194.

[4] Cited in Calvin C. Clawson, *Mathematics Mysteries* (Cambridge, Massachusetts: Perseus Books, 1996), p. 17. Peano actually started with the number zero instead of the number 1.

Axiom 1. 1 is a number.

Axiom 2. The successor of any number is a number.

Axiom 3. No two numbers have the same successor.

Axiom 4. 1 is not the successor of any number.

Axiom 5. If 1 has a certain property, and the successor of every number has the same property, then every number has that property.

Let's see if we can determine what words Peano considered to be foundational and therefore *undefined*. There are three. Think about it for a moment.

Here are the three foundation assumptions: (1) number, (2) the number 1, and (3) successor. These terms are assumed and therefore left undefined because we have an intuitive sense for them; i.e., they are derived due to of the image of God in man. When we combine Axiom 1 with Axiom 4, we can conclude that 1 is the first number of the natural numbers. Axiom 2 states that the successor of any number is a number. This guarantees that the sequence of counting numbers *always* contains numbers. Axiom 3 guarantees every number in the sequence is one greater than the previous number (it gives us the pattern) and that every number in the sequence is unique. Hence, we will not get something like this: {1, 2, 3, 2, 4, 5, 3, 6, 2, …}. We will wait a few moments before we discuss Axiom 5.

Note also that in these axioms we have encapsulated or subsumed the set of natural numbers (an *infinite* set) by a series of *finite* statements (i.e., the infinite, in some ways, can be subsumed by the finite). These axioms are also *sufficient* to define all the operations of arithmetic. Let's see how this works. First, consider addition. What is 3 + 4? Using Peano's axioms:

Step 1.	Write 4 as a successor of 3:	3 + (3 + 1)
Step 2.	Write the second 3 as a successor of 2:	3 + (2 + 1 + 1)
Step 3.	Rearrange[5] the 1s:	(3 + 1) + (2 + 1)
Step 4.	Write the successor of 3 as 4	4 + (2 + 1)
Step 5.	Write the 2 as a successor of 1	4 + (1 + 1 + 1)
Step 6.	Rearrange the 1s:	(4 + 1) + (1 + 1)
Step 7.	Write the successor of 4 as 5	5 + (1 + 1)
Step 8.	Rearrange the 1s:	(5 + 1) + 1
Step 9.	Write the successor of 5 as 6	6 + 1
Step 10.	Write the successor of 6 as 7	7

Fortunately, we do not have to go through all that trouble just to determine that 3 + 4 = 7. The purpose of this demonstration is to show how Peano's axioms can account for (or justify) the process of addition which is really a method of counting.

Multiplication can be defined as repeated addition. That is, 4 × 3 = 3 + 3 + 3 + 3 or 4 + 4 + 4 = 12. *Multiplication is repeated addition by the same number* or, in symbols:

$$b \times a = \underbrace{a + a + \cdots + a}_{\text{b times}}$$

[5] Rearranging numbers like this is called the *associative property* of addition.

Exponentiation can be defined as repeated multiplication. That is, $4^3 = 4 \times 4 \times 4 = 64$. *Exponentiation is repeated multiplication by the same number* or, in symbols:

$$a^b = \underbrace{a \times a \times \cdots \times a}_{\text{b times}}$$

Subtraction and division are inverse operations of addition and multiplication respectively. In the subtraction problem, 9 – 3, we simply ask the question, "What must be added to 3 to get 9?" In the division problem, $12 \div 3$ or $\frac{12}{3}$, we ask the question, "What must be multiplied by 3 to get 12?"

What is fascinating and wondrous to note is that tumbling out of Peano's axioms defining the infinite set of natural numbers are *all the characteristics of the rudimentary operations of arithmetic.*

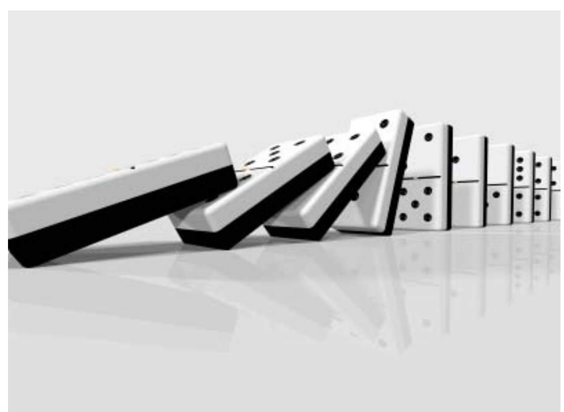
Now to Axiom 5, the explication of which, is the main point of this essay. Axiom 5 allows us to prove things about *all* numbers in the natural number sequence *even though there are an infinite number of them.* Axiom 5 is the definition of *mathematical induction* or "reasoning by recurrence."[6] The power of mathematical induction is that by it, we are able to prove universal statements about specific patterns involving the set of *infinite* natural numbers.

The earliest evidence of the use of mathematical induction is found in Euclid's proof that the number of primes is infinite. The earliest implicit proof by mathematical induction for arithmetic sequences was introduced by an Arabic mathematician, al-Karaji (ca. 1000 AD). Shortly afterwards, other Arabic mathematicians followed al-Karaji's lead.

None of these ancient mathematicians, however, *explicitly* stated the two steps of induction. The first rigorous explicit exposition of the principle of induction was given by Francesco Maurolico (1494-1575), in his *Arithmeticorum libri duo* (1575), who used the technique to prove that the sum of the first *n* odd natural numbers is n^2 (we shall do that shortly). The inductive hypothesis was also discovered independently by the Swiss mathematician Jakob Bernoulli (1654-1705), and the French mathematicians Blaise Pascal (1623-1662) and Pierre de Fermat (1601-1665).

I like to call mathematical induction the "domino principle." If you have played with dominoes, you probably lined up several on end. Then, you knocked the first one down just to see the whole set topple in succession.

What is the explicit procedure of mathematical induction? The *modus operandi* is summarized in Axiom 5 and it needs to be unpacked "step by step."



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Suppose we wish to prove an assertion or a conjecture about the natural numbers. We let A(n) denote this assertion regarding *any* natural number *n.* Note how we have generalized the situation. Instead of looking at a specific natural number (like 5 or 789), we consider *any* natural number and we represent this natural number using the letter *n.* To prove that A(n) is true for all *n* (the infinite set of "dominoes"), it suffices to prove two things:

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[6] Mathematical induction is *not* the same as inductive thinking. Inductive thinking is finding general patterns given particular instances of something. It is the "bread and butter" method of scientific empiricism. In contrast, deduction is a way of thinking that starts from given axioms and definitions. Using these generalities, you can then use the laws of logic to justify particular statements. Mathematical induction is a form of deduction applied primarily to the set of natural numbers. The phrase "reasoning by recurrence" better encapsulates the logical process.

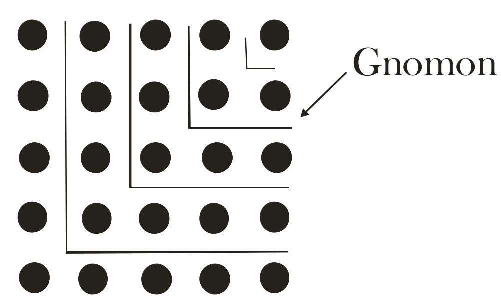
Step 1 (step of induction). We must show the A(1) is true; i.e., the assertion is true when n = 1. Using our domino illustration, this is equivalent to toppling the first domino.

Step 2 (step of inheritance). We next assume that the assertion is true for all natural numbers less than or equal to *n* (in symbols, $\leq$ n). Then, all we have to do is prove that this assertion is true for A(n + 1). If we can do this, then the initial assertion, A(n) is true. Using our domino illustration, this is equivalent to the toppling of all the dominoes.

These two steps comprise the logical process called mathematical induction. Both steps, the induction step and the hereditary step, are necessary; neither is sufficient alone. Step 1 gives us a starting point (we knock down the first domino). Step 2 shows us that we can prove A(2) from A(1) or the first domino has knocked down the second, then A(3) from A(2) or the second domino has knocked down the third, etc., *ad infinitum*. Stunning logic, isn't it?

Let's now apply our methods to some examples. First, notice the following pattern.

$1 = 1\cdot1 = 1^2$ (n = 1; i.e., how many numbers you are summing)
$1 + 3 = 4 = 2\cdot2 = 2^2$ (n = 2)
$1 + 3 + 5 = 9 = 3\cdot3 = 9 = 3^2$ (n = 3)
$1 + 3 + 5 + 7 = 16 = 4\cdot4=16 = 4^2$ (n = 4)



The ancient Greeks were the first to notice this pattern and they noticed it using geometry or arranging pebbles in squares. In general, it appears that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$. This is our assertion, our A(n). Can we prove that this pattern is always true?

Step 1. First, is it true when n = 1 or for A(1)? Yes, when n = 1, then $1 = 1\cdot1 = 1^2 = 1$.

Step 2. We shall now *assume* it is true for some *n* in general. We assume that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ is actually true. We ask, "Does this pattern now hold for the successor of n, i.e., n + 1 or A(n + 1)?"

Given $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, we need to add 2n + 1 to both members of this equation. We get:

$1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = n^2 + 2n + 1$

We note two things:

$2n + 1 = 2(n + 1) - 1$
$n^2 + 2n + 1 = (n + 1)^2$

Substituting, we get: $1 + 3 + 5 + \ldots + (2n - 1) + [2(n + 1) - 1] = (n + 1)^2$. Hence, what is true for *n* is also true for *n + 1* or A(n + 1) is true. In other words, the sum of the first *n* odd numbers is n^2 and the sum of the first (n + 1) odd numbers is $(n + 1)^2$. Therefore, by mathematical induction, the sum of all odd numbers up to 2n – 1 will *always equal* n^2. QED.

Let's investigate a few more. The German mathematician (1777-1855) Carl Friedrich Gauss, as a young boy, determined that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. This pattern, our A(n) is called the Gaussian sum is his honor. Let's prove this pattern using mathematical induction.

Step 1. For n = 1, $\frac{1(1+1)}{2} = 1$. Therefore, A(1) is true.

Step 2. We assume that A(n) is true for n; i.e., 1 + 2 + 3 + … + n = $\frac{n(n+1)}{2}$. Then, for n + 1, we get:

$$1 + 2 + \ldots + n + (n + 1) = \frac{n(n+1)}{2} + (n + 1) =$$
$$(n+1)\left(\frac{n}{2}+1\right) =$$
$$\frac{(n+1)(n+2)}{2} =$$
$$\frac{(n+1)((n+1)+1)}{2}$$

Hence, we have established the truth of A(n +1). Do you see it? Therefore, by mathematical induction, A(n) is true for any natural number *n*. QED.

Let's now prove A(n): $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

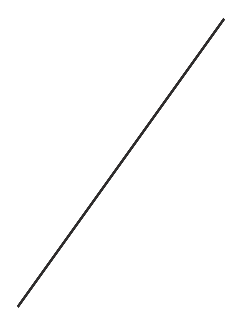
Step 1. For n = 1, $\frac{1(1+1)(2\cdot1+1)}{6} = \frac{2\cdot3}{6} = 1 = 1^2$. Therefore, A(1) is true.

Step 2. We suppose that A(n) is true for n; i.e., $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. Then, for n + 1. we get"

$$1^2 + 2^2 + \ldots + n^2 + (n + 1)^2 = \frac{n(n+1)(2n+1)}{6} + (n + 1)^2 =$$
$$(n+1)\left(\left(\frac{n(2n+1)}{6}\right)+(n+1)\right) = (n+1)\left(\frac{n(2n+1)+6(n+1)}{6}\right) =$$
$$(n+1)\left(\frac{2n^2+n+6n+6}{6}\right) = (n+1)\left(\frac{2n^2+7n+6}{6}\right) =$$
$$(n+1)\left(\frac{(n+2)(2n+3)}{6}\right) = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

Hence, we have established the truth of A(n +1). Therefore, by mathematical induction, A(n) is true for any natural number n. QED.

For a final example, let's apply mathematical induction to geometry. We assert A(n): *n* distinct concurrent lines (concurrent lines intersect in one point) in a plane divide the plane into *2n* regions.



Step 1

Step 1. Since one line in a plane divides the plane into two parts, then A(1) is true.

Step 2. We suppose that A(n) is true for *n*: *n* lines divide their plane into *2n* parts. By drawing one more line (n + 1 lines), concurrent with these *n* lines, we shall divide two vertical angles into 4 angles, and thus 2 more angles will be added. Now the plane will be divided into 2n + 2 = 2(n + 1) parts. Hence, A(n + 1) is true, and, by mathematical induction, we have proved A(n) for an arbitrary number of lines. QED.

As stated before, mathematical induction is a very powerful form of logic. Mathematicians have used it to justify the truth of a wide variety of mathematical propositions *if these propositions can be reformulated using the set of natural numbers*. The logic of the toppling of dominoes applied to counting has sweeping consequences. Who would ever have thought of these ramifications when they first played with dominoes or first counted using their fingers?