Why does i^2 = -1?

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Introductory comments

Many students express curiosity when they encounter imaginary numbers, or, more technically, the complex number system.[1] They wonder why in the world one can work with $\sqrt{-1}$, an impossibility, because, in the real number system, no number multiplied by itself equals -1. Many mathematicians in 16th century Europe thought likewise. Contrariwise, some engaged in the following proposition, "Even though we know that the answer to $\sqrt{-1}$ is nonsensical, let's experiment with it and see what happens." This mathematical experiment, in time, proved to have a multiplicity of real world applications, quantum mechanics and the electrical theory of alternating current being two prominent examples.

Historically, square roots of negative numbers appeared in the solution of quadratic equations.[2] The following analysis explores the nature of "imaginary" numbers in this context.

Analysis

We consider the equation $x^2 + 1 = 0$. Solving for x, we get $x^2 = -1$.

Taking the square root of both members of this equation, we get:

$x = \sqrt{-1}$ (which creates our problem)

Although we have no idea what $\sqrt{-1}$ is equal to, we arbitrarily let it be equal to i (i standing for "imaginary"). If $i = \sqrt{-1}$, weapply some algebra to find out the equivalent of i^2.

$$i = \sqrt{-1} \Leftrightarrow i^2 = \left(\sqrt{-1}\right)\left(\sqrt{-1}\right)$$

1) It would seem reasonable to conclude that, since $\left(\sqrt{a}\right)\left(\sqrt{b}\right) = \sqrt{ab}$,

$$i^2 = \left(\sqrt{-1}\right)^2 = \left(\sqrt{-1}\right)\left(\sqrt{-1}\right) = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

This conclusion creates a problem. Note carefully this algebraic argument:

$$i^2 = \left(\sqrt{-1}\right)^2 = \left(-1^{\frac{1}{2}}\right)^2$$

2) Since $\left(a^m\right)^n = a^{mn}$, then $i^2 = \left(-1^{\frac{1}{2}}\right)^2 = -1$!

By 1) and 2), we conclude that 1 = -1, a contradiction! We must resolve this contradiction. Either $i^2 = -1$ or $i^2 = 1$. Which is it?

We must conclude that $\left(\sqrt{-1}\right)\left(\sqrt{-1}\right) = \sqrt{(-1)(-1)}$ *is an invalid equality*. Why? Note what we started with, the equation: $x^2 = -1$. From this we get: $x = \sqrt{-1}$ and we let $i = \sqrt{-1}$. Hence, *it is required* that $i^2 = -1$ and not that $i^2 = 1$. Hence, $i \times i = i^2 = -1$. QED.

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[1] A complex number is of the form a + bi where a, b are real numbers (real numbers are either rational or irrational numbers) and $i = \sqrt{-1}$.

[2] Quadratic equations are of the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$.

From this analysis, we can generate a list of sequential powers of i. This list forms the basis for the arithmetic of complex numbers.

$i =$	$\sqrt{-1}$
$i^2 = (\sqrt{-1})(\sqrt{-1}) =$	-1
$i^3 = (\sqrt{-1})(\sqrt{-1})(\sqrt{-1}) = -1(i) =$	-i
$i^4 = (\sqrt{-1})(\sqrt{-1})(\sqrt{-1})(\sqrt{-1}) = (-1)(-1)$	+1

Any larger power of *i* can be reduced to one of these basic four. For example:

$i^5 = i^{4+1} = i^4 i^1 = (1)(\sqrt{-1}) = (\sqrt{-1}) = i$
$i^{15} = i^{4+4+4+3} = i^4 i^4 i^4 i^3 = (1)(1)(1)(-i) = -i$