Geometric Sequences

by James D. Nickel

In an arithmetic sequence, a common "difference" separates each term in the sequence. For example, in the sequence 5, 10, 15, 20, 25, …, the common difference between terms is 5. In a *geometric sequence*, you calculate each successive term by *multiplying by the same number*. Instead of a common difference separating terms, there is or there is a *common ratio* or *common multiplier* between each pair of adjacent terms in the sequence. For example, in the sequence $1/10, 1/100, 1/1000, 1/10,000, \ldots$, the common multiplier or ratio is $1/10$. In the sequence 5, 10, 20, 40, 80, …, the common multiplier is 2. Written as a ratio, $2 = 2/1$.

Ratio is another way to view a fraction. For example, $1/4$, as a fraction, means part of a whole or, more specifically, one part of four parts of a whole. If the whole is a piece of pie, $1/4$ means one-fourth of the pie. $1/4$ is also a division problem (meaning "1 divided by 4"). As a decimal, the quotient is 0.25 = $25/100 = 1/4$! A ratio is a type of comparison between two objects having the same unit. For example, $1/4$, as a ratio, means that 1 car in 4 in the parking lot is a Toyota (unit = cars). Or, 1 in every 4 people have brown eyes (unit = eyes).

Here are some examples of geometric sequences (along with their common multiplier or ratio):

4, 12, 36, 108, 324, … (common ratio or multiplier is $3/1$)

5, 20, 80, 320, … (common ratio or multiplier is $4/1$)

6, 4, $8/3$, $16/9$, … (common ratio or multiplier is $2/3$)

1, $1/2$, $1/4$, $1/8$, … (common ratio or multiplier is $1/2$)

9, 3, $1/3$, $1/9$, …(common ratio or multiplier is $1/3$)

Notice that all of these examples are examples of *infinite* geometric sequences (… means continue the process *ad infinitum*). The three basic units of time (hours, minutes, seconds) comprise a *finite* geometric sequence.

- 1 hour equals 60 minutes.
- 60 seconds equals 1 minute (60 × 1).
- 3600 seconds in one hour (60 × 60).

The sequence: 1, 60, 3600. The common multiplier is 60 or, as a ratio, $60/1$.

We can query about the sum of both arithmetic and geometric sequences. The sum of a geometric sequence is called a *series*. Rational numbers that have repeating decimals are sums of a geometric series

In these examples, note especially that the *infinite* sums on the right hand side of the equation equal a *finite* number.[1]

$$\frac{1}{3} = 0.\bar{3} = 0.3333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \ldots$$

[1] Technically, the sum of an infinite series *converges* to a finite number as a limit. The … (*ad infinitum*) tells you that the series continues "without end."

$$\frac{2}{3} = 0.\bar{6} = 0.6666\ldots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10,000} + \ldots$$

$$\frac{1}{9} = 0.\bar{1} = 0.1111\ldots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10,000} + \ldots$$

The common multiplier or ratio in these three examples is $1/10$. We can draw a fascinating conclusion from the first two examples. Watch carefully!

$$\frac{1}{3} = 0.\bar{3} = 0.3333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \ldots$$

$$\frac{2}{3} = 0.\bar{6} = 0.6666\ldots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10,000} + \ldots$$

$$\frac{1}{3} + \frac{2}{3} = 0.\bar{3} + 0.\bar{6} = 0.\bar{9} = 1 \text{ or}$$

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \ldots + \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10,000} + \ldots =$$

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10,000} + \ldots = 1$$

Ponder that for a while!

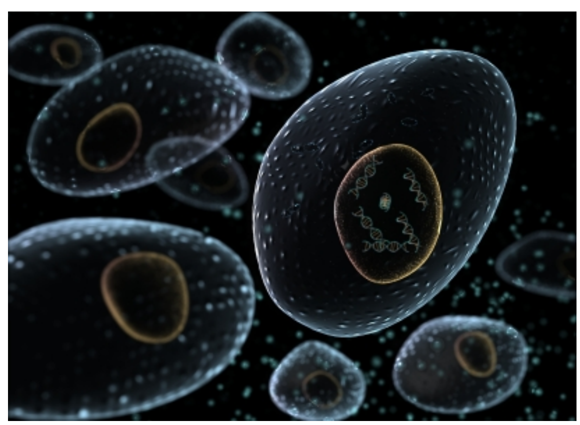
A *binary sequence* is a special type of geometric sequence. It looks like this: 1, 2, 4, 8, 16, … (where the common ratio or multiplier is $2/1$). This sequence is also called a "doubling" sequence. Recall that raising to a power is repeated multiplication. For example, $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$. In general, any number can be raised to an integral power. If we let *b* be equal to any number and *a* be a positive integer, then $b^a = \underbrace{b \times b \times \ldots \times b}_{\text{a times}}$. As powers, we can write the binary sequence as follows:

$$1, 2, 4, 8, 16, \ldots = 2^0, 2^1, 2^2, 2^3, 2^4, \ldots$$

The nature of the doubling sequence (1, 2, 4, etc.) requires the following definitions. Any number raised to the power of 0 is 1; i.e., $a^0 = 1$. Any number raised to the power of 1 is that number; i.e., $a^1 = a$.

The growth of an organism by cell division replicates the binary sequence:

- 1 cell = $2^0 = 1$
- 2 cells = $2^1 = 2$
- 4 cells = $2^2 = 2 \times 2$
- 8 cells = $2^3 = 2 \times 2 \times 2$
- 16 cells = $2^4 = 2 \times 2 \times 2 \times 2$



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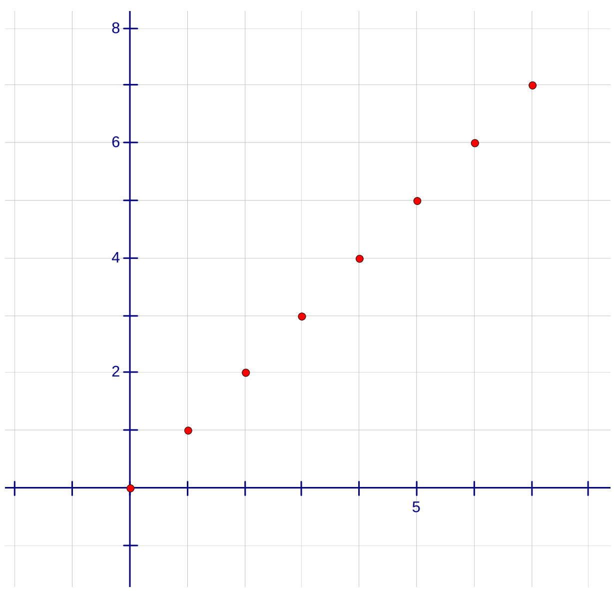
A good way to compare the difference between an arithmetic sequence (like the counting numbers) and a geometric sequence (like the binary sequence) is to picture these sequences in graphical form.

First, we construct two tables:

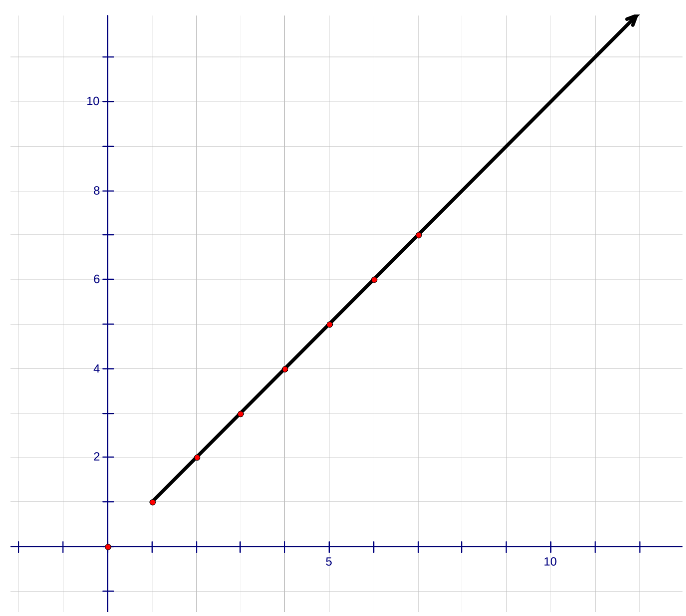
Term	*Number*	*Rate of Change (Common Difference)*
1st	1	-
2nd	2	1
3rd	3	1
4th	4	1
5th	5	1
⋮	⋮	⋮

Term	*Number*	*(Common ratio)*	*Difference*
1st	1	-	-
2nd	2	2	1
3rd	4	2	2
4th	8	2	4
5th	16	2	8
⋮	⋮	⋮	⋮

Second, we plot these points on a Cartesian coordinate plane. If we *connect* the points of the arithmetic sequence, we get a straight line. The graph of an arithmetic sequence will always be a straight line. In this example, the equation of the graph is $y = f(x) = x$. Since the domain of this function is $x \in +\mathbb{Z}$ (meaning the set of "positive integers" or counting numbers[2]), we cannot technically draw the line between the points. We do so for the sake of illustration.



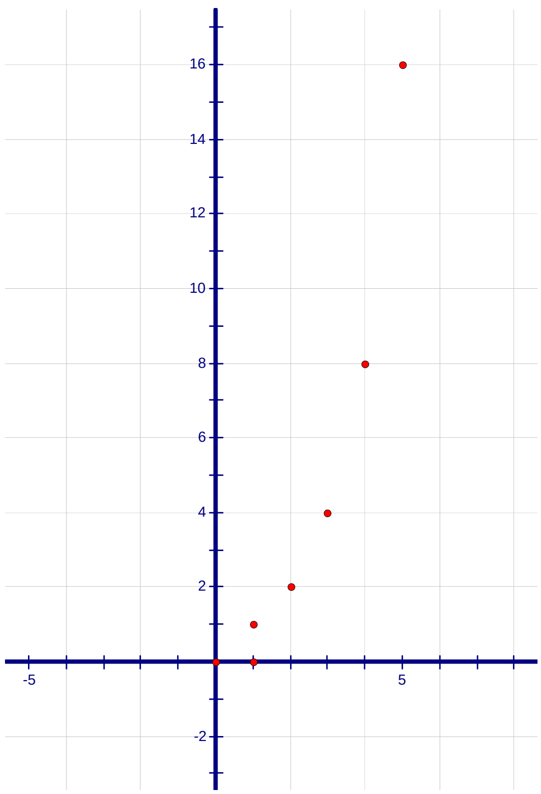
Arithmetic Sequence



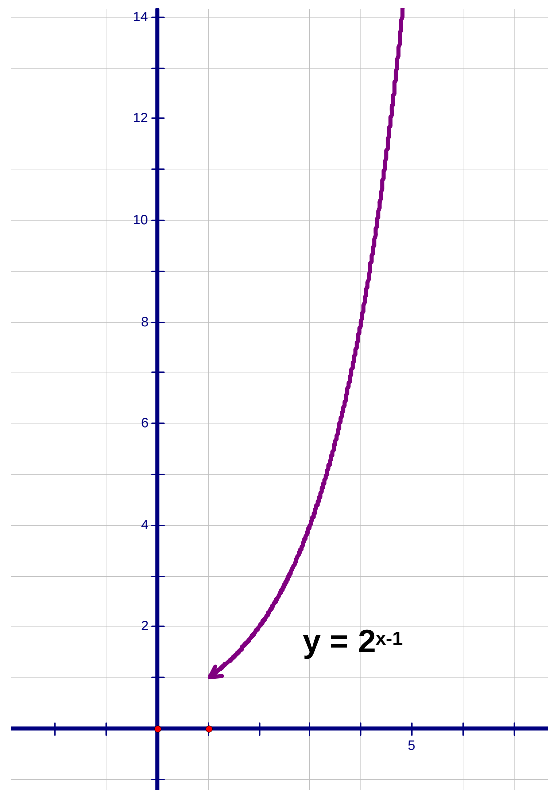
Straight Line

[2] The mathematical symbol for the set of counting numbers (also called natural numbers) is ℕ.

If we connect the points of the geometric sequence, we get a curved line. The graph of an geometric sequence will always be a curved line. In this example, the equation of the graph is $y = f(x) = 2^{x-1}$. Again, since the domain of this function is $x \in +\mathbb{Z}$, we cannot technically draw the line between the points. We do so again for the sake of illustration. The graph illustrates the remarkable growth of the binary or doubling sequence.



Geometric (Binary) Sequence



Curved Line

The binary sequence is also contained in the binary number system. This number system, used as the technological backbone of integrated computer circuits, has only two digits, 0 and 1. Since these two digits represent the presence or absence of electrical current in a circuit, the binary system "fits" like a glove. Counting in the binary system is just like counting in the decimal system. The only difference is the change in base. Note especially the binary expansion of numbers written in base 2 in the following table:

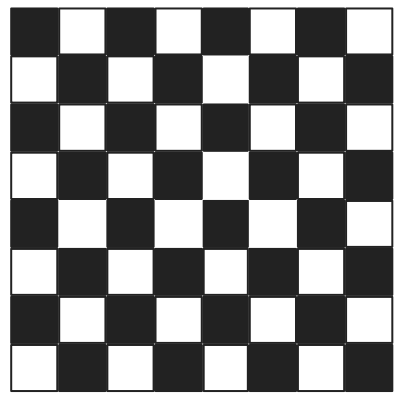
Base 10	*Base 2*	*Binary Expansion*	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
1	1	1×2^0					1
2	10	$1 \times 2^1 + 0 \times 2^0$				1	0
3	11	$1 \times 2^1 + 1 \times 2^0$				1	1
4	100	$1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$			1	0	0
5	101	$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$			1	0	1
6	110	$1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$			1	1	0

GEOMETRIC SEQUENCES

BY JAMES D. NICKEL

Base 10	*Base 2*	*Binary Expansion*	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
7	111	$1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$			1	1	1
8	1000	$1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$		1	0	0	0
9	1001	$1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$		1	0	0	1
10	1010	$1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$		1	0	1	0
11	1011	$1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$		1	0	1	1
12	1100	$1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$		1	1	0	0
13	1101	$1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$		1	1	0	1
14	1110	$1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$		1	1	1	0
15	1111	$1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$		1	1	1	1
16	10000	$1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$	1	0	0	0	0

In conclusion, the binary sequence is revealed in a famous legend from antiquity. Consider King Shirham of India wanted to reward his grand vizier Sissa Ben Dahir for inventing and presenting to him the game of chess. The clever vizier responded as he knelt before the king:



> "Majesty, give me a grain of wheat to put on the first square of this chessboard, and two grains to put on the second square, and four grains to put on the third, and eight grains to put on the fourth. And so, oh King, doubling the number for each succeeding square, give me enough grains to cover all 64 squares of the board."

Kneading a bag of wheat with his fingers, the King replied, "You do not ask for much, oh my faithful servant. Your wish will certainly be granted."

How many grains of wheat will the King need? The answer to the problem consists in summing the powers of two: 2^0 (1 in the first square) + 2^1 (2 in the second square) + 2^2 (4 in the third square) + … + 2^{63} (in the 64^{th} square).

Do you think the bag of wheat in the King's hand is enough?

How do we find this sum? We need a formula! In general, we can write a geometric series as follows where a = our starting number and r = the common multiplier or ratio:

$$ar^0 + ar^1 + ar^2 + ar^3 + \ldots + ar^n$$

In our example, a = 1 and r = 2. If we substitute these numbers into this general form, we get our series:

$$2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^n$$

Let's continue working with the general form, though. Using sigma notation (Σ), we can shorten the general formula and let S = its sum:

$$S = \sum_{i=0}^{n} ar^i = ar^0 + ar^1 + ar^2 + ar^3 + \ldots + ar^n$$

Multiply both sides by 1 – r (Note: $r \neq 1$). Why? We will find out shortly. We get:

$$S(1-r)=(1-r)\left(ar^0+ar^1+ar^2+ar^3+\ldots+ar^n\right)$$

Simplifying the right side of the equation, we get:

$$S(1-r)=ar^0+ar^1+ar^2+ar^3+\ldots+ar^n-\left(ar^1+ar^2+ar^3+\ldots+ar^n+ar^{n+1}\right)$$

Note the terms that “drop out” on the right side (this is why we multiplied by 1 – r). We get, in sequence:

$$S(1-r)=ar^0+ar^1+ar^2+ar^3+\ldots+ar^n-ar^1-ar^2-ar^3-\ldots-ar^n-ar^{n+1}$$

$$S(1-r)=ar^0\cancel{+ar^1}\cancel{+ar^2}\cancel{+ar^3}\cancel{+\ldots}\cancel{+ar^n}\cancel{-ar^1}\cancel{-ar^2}\cancel{-ar^3}\cancel{-\ldots}\cancel{-ar^n}-ar^{n+1}$$

$$S(1-r)=ar^0-ar^{n+1}$$

Multiplying both sides by -1, we get:

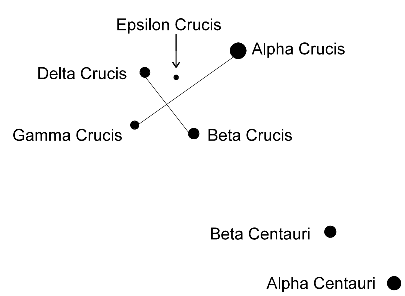
$$S(r-1)=ar^{n+1}-ar^0$$

Dividing both sides by r – 1 (Note again, $r \neq 1$, or else we are dividing by zero), noting that $r^0 = 1$, and doing some factoring, we get our formula[3]:

$$S=\frac{ar^{n+1}-ar^0}{r-1}=\frac{a\left(r^{n+1}-1\right)}{r-1}$$

We want to sum $2^0 + 2^1 + 2^2 + \ldots + 2^{63}$. Again, we note a = 1, r = 2, n = 63. Plugging in the numbers, we get $S = \dfrac{a\left(r^{n+1}-1\right)}{r-1}=\dfrac{2^{64}-1}{2-1}=2^{64}-1=18,446,744,073,709,551,615\approx 1.84\times 10^{19}$

This number of grains would be enough to cover the *entire surface of the earth* with a layer of wheat one-half inch in height! This many grains would be equivalent, in weight, to about 175 billion *tons!* If we placed that many grains in an unbroken line, the line would be two light-years in length. A light-year is defined as the distance light travels in one year (light travels at a speed of about 186,000 miles per second or 300,000 kilometers per second). The distance of two light-years is about half the distance from Earth to the nearest star beyond our solar system, Alpha Centauri.



The King seriously “underestimated” the vizier’s extravagant request! According to the legend, since the King could not grant this request, he ordered his court to lop off the poor vizier’s head. The lesson learned: “It is not too wise, for your head, that is, to be too clever!”

[3] This formula is the basis for determining the sum of an *infinite* geometric series. All you need to do is consider what happens to the formula for different values of r ($0 < r < 1$, $r = 1$, and $r > 1$) when *n* gets very, very large (or, using mathematical symbols, as $n\to\infty$).