7.5 Multiplying Decimals

Measurement Review and Extension

Terms & Symbols Introduced
1. ±
2. Calibration
3. Density
4. Meniscus

Knowing the number of significant figures is critical in scientific measurement. Every measuring tool has a level or degree of precision and this precision always allows for one estimated digit.

The volume of an object is measured in the chemistry laboratory using different glassware, all metric. The most common analog tools are the (1) beaker, (2) graduated cylinder, and (3) buret.

Which glassware gives the most precise volume measurement? Let's investigate each to find out its degree of precision or its boundary of error (Lesson 7.3). Doing this will provide us with the number of significant figures.

> Metric System (Base 10) Principle: Determine the smallest division of the measuring tool and read the volume from the analog scales to one tenth or 0.1 of the smallest division.[1] Therefore, the error in reading, or the reading error, is one tenth or 0.1 of the smallest division on the glassware.

Beaker

In Figure 1, the smallest division is 10 mL, so we can read the volume to 1/10 of 10 mL, or 1 mL. If we read the volume as k mL, there will be a read error, and it will be between $(k - 1)$ mL and $(k + 1)$ mL. Scientists use the **plus or minus symbol** ±, indicating the operation of addition or subtraction, to signify these boundary conditions (Lesson 5.6). The volume we read from the beaker has a reading error, therefore, of ± 1 mL.

The volume in this beaker is 52 mL ± 1 mL. You might have read the volume as 51 mL. Another person might read the volume as 53 mL. All the answers are correct within the reading error of ± 1 mL. If our measure is 52 mL, the exact measure will have boundary conditions. It will be somewhere between 51.5 mL and 52.4 mL.

How many significant figures does 52 mL have? 2. We know the "5" digit for certain, but we had to estimate the "2" digit.

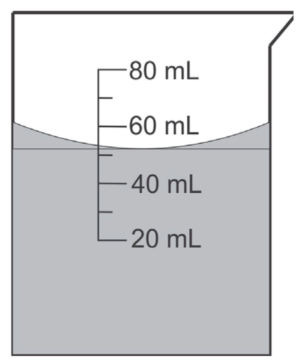

Figure 1. Beaker.

Graduated Cylinder

The smallest division of this graduated cylinder (Figure 2) is 1 mL. Therefore, our reading error will be ± 0.1 mL since 1/10 of 1 mL is 0.1 mL.[2] A possible reading of the volume is 16.8 mL ± 0.1 mL. An equally precise measurement would be 16.9 mL or 16.7 mL. If our measure is 16.8 mL, the exact measure will be somewhere between 16.75 mL and 16.84 mL.

How many significant figures does 16.8 mL have? 3. We know the "1" and the "6" digit for certain, but we had to estimate the "8" digit.

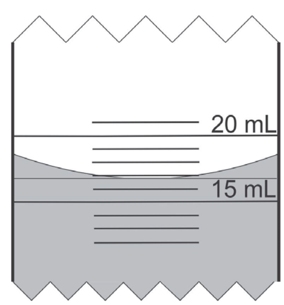

Figure 2. Graduated Cylinder.

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[1] This principle applies to any measurement tool, e.g., rulers, calipers, etc. You just must be aware of the base.

[2] Depending upon the type of graduated cylinder used in measurement, the surface of the liquid might be curved ◡. Named **meniscus**, from a Greek word meaning "crescent," this curvature is caused by water molecules being attracted more to glass

BURET

The smallest division in this buret (Figure 3) is 0.1 mL. Therefore, our reading error is ± 0.01 mL since 1/10 of 0.1 mL is 0.01 mL.[3] A possible volume reading is 40.26 mL ± 0.01 mL. An equally precise measurement would be 40.25 mL or 40.27 mL. If our measure is 40.26 mL, the exact measure will be somewhere between 40.255 mL and 40.264 mL.

How many significant figures does 40.26 mL have? 4. We know the "4", "0", and "2" digits for certain, but we had to estimate the "6" digit.

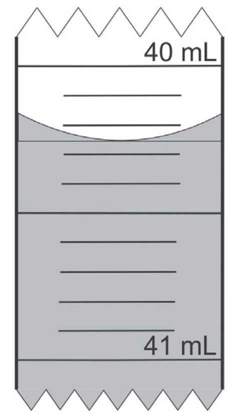

Figure 3. Buret.

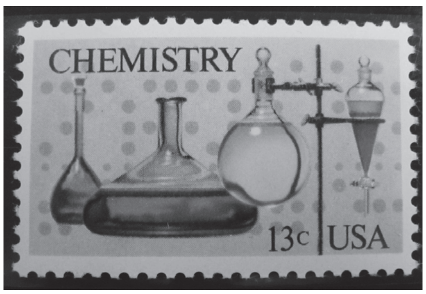

Figure 4. The glassware of Chemistry. Source: United States Postal Service.

CONCLUSIONS

We can link the number of significant figures to precision of measurement. If we only need a rough estimate of volume, we use the beaker (two significant figures). If we want better precision, we should use the graduated cylinder (three significant figures) or the buret (four significant figures).

The concept of significant figures has a relationship to precision, not accuracy. Think about measuring the length of an object with a ruler in base 2 divisions. Then, measure the same object again with the same ruler five more times. Would the measurements be different? We will get the same measurement repeatedly with a small reading error equal to about one-half of the smallest division on the ruler. We have determined the length with precision, not accuracy. Precision is concerned with the reproducibility of a measurement. Precision depends upon the skill of the person making the measurement.

We can now revisit and expand Table 1 of Lesson 7.3. Note the dance between column two and column three; i.e., the product of these numbers is always 1.

Table 1: Measurement Reading Error

Analog Tool	Division Between Units	Smallest Division	Base	Reading Error
Inch Ruler	2 equal parts	$\frac{1}{2}$ inch	2	$\frac{1}{2}$ of $\frac{1}{2} = \pm\frac{1}{4}$ inch
Inch Ruler	4 equal parts	$\frac{1}{4}$ inch	2	$\frac{1}{2}$ of $\frac{1}{4} = \pm\frac{1}{8}$ inch
Inch Ruler	8 equal parts	$\frac{1}{8}$ inch	2	$\frac{1}{2}$ of $\frac{1}{8} = \pm\frac{1}{16}$ inch
Inch Ruler	16 equal parts	$\frac{1}{16}$ inch	2	$\frac{1}{2}$ of $\frac{1}{16} = \pm\frac{1}{32}$ inch
Inch Ruler	32 equal parts	$\frac{1}{32}$ inch	2	$\frac{1}{2}$ of $\frac{1}{32} = \pm\frac{1}{64}$ inch

than to each other. (Note: Technically, adhesive forces are stronger than cohesive forces.) We read the volume on the scale at the *bottom* of the meniscus.

[3] Note that the numbers get greater as you go down the buret, an arrangement different from the beaker or the graduated cylinder. This increase from top to bottom is because the liquid leaves the buret at the bottom. Refer to the picture of the buret in Lesson 7.3.

Table 1: Measurement Reading Error

Analog Tool	Division Between Units	Smallest Division	Base	Reading Error
Inch Ruler	64 equal parts	$\frac{1}{64}$ inch	2	$\frac{1}{2}$ of $\frac{1}{64} = \pm\frac{1}{128}$ inch
Inch Dial Caliper	1000 equal parts	$\frac{1}{1000}$ inch	10	$\frac{1}{10}$ of $\frac{1}{1000} = \pm\frac{1}{10{,}000}$ inch
Metric Beaker	$\frac{1}{10}$ equal part	10 mL	10	$\frac{1}{10}$ of $10 = \pm 1$ mL
Metric Graduated Cylinder	1 equal part	1 mL	10	$\frac{1}{10}$ of $1 = \pm\frac{1}{10}$ mL
Metric Buret	10 equal parts	0.1 mL	10	$\frac{1}{10}$ of $0.1 = \pm\frac{1}{100}$ mL

How do we know a ruler is accurate? A plastic ruler left in the hot sun would stretch, generating inaccurate measurements. The only way a scientist can guarantee accuracy is by **calibration**.[4] To calibrate a ruler, we would compare it against a ruler we know is accurate. We compare the tool with a standard.[5] In the chemistry lab, the scientist would calibrate volumetric glassware before using it by weighing a known volume of liquid in the glassware. By dividing the mass of the liquid by its **density**,[6] the scientist can determine the actual volume and therefore the accuracy of the glassware.

> Accuracy is how close a measurement comes to the actual measurement.
>
> Precision is concerned with the reproducibility of a measurement.
>
> In measurements, you can be precise but not accurate. You could also be accurate but not precise.
>
> A way to understand accuracy and precision is the game of darts. If a player is accurate with his dart throws, his aim will always take the dart close to the center of the circular dartboard. If the player is precise with his throws, his aim will always take the dart to the same location, but this location may be close to the center or far away from it. A good player will be both accurate and precise. This player will throw the dart the same way each time and each time his dart will hit near the center of the dartboard.

[4] It is assumed that the measuring devices scientists use are accurate according to the specifications given by the manufacturer of that device.

[5] As a principle, Scripture requires just weights; i.e., accurate weights (Lesson 6.14), to curtail cheating. Hosea 12:7, "A merchant, in whose hands are false [deceitful] balances, he loves to oppress [defraud]."

[6] Density D is defined as the mass M of an object divided by its volume V or $D = M/V$. We measure V of a liquid known to have a density D. Then, we weigh the mass M_1 of this volume. Next, we do these algebraic operations: $D = M/V \Leftrightarrow DV = M \Leftrightarrow V = M/D$. (See Step 8 for more of this type of work.) We divide M_1 by D to get V_1 and compare V_1 to V to determine the accuracy of the glassware.

THE METHOD OF MULTIPLYING DECIMALS

> ... computing is, or at least should be, intimately bound up with both the source of the problem and the use that is going to be made of the answers – it is not a step to be taken in isolation from reality.
>
> R. W. Hamming, *Numerical Methods for Scientists and Engineers* ([1962] 1973), p. 3.

Let's explore how to multiply decimals using an area example.
Example 1. What is the area of a rectangular garden plot 4.25 yards by 6.75 yards?
Estimate:

$$4 \times 7 = 28$$

We can find the exact product in two ways:

Method 1. We convert the decimal to fractions:

$$4.25 = 4\frac{\overset{1}{\cancel{25}}}{\underset{4}{\cancel{100}}} = 4\frac{1}{4} = \frac{17}{4}$$

$$6.75 = 6\frac{\overset{3}{\cancel{75}}}{\underset{4}{\cancel{100}}} = 6\frac{3}{4} = \frac{27}{4}$$

Multiplying, we get:

$$\frac{27}{4} \times \frac{17}{4} = \frac{459}{16} = 28\frac{11}{16} \text{ because:}$$

$$16 \quad 4\,{}_{2}\overset{13}{5}\,{}_{5}\overset{11}{9}$$
$$\quad\quad 2\ 8$$

How do we convert this mixed number back to a decimal? We know that $28\frac{11}{16}$ means $28 + \frac{11}{16}$ and $\frac{11}{16}$ means 11 divided by 16. Let's do the division, a portend of Lesson 7.6:

$$16 \quad 11$$

When the divisor is greater than the dividend, we insert the decimal point after the ones position in the dividend, add more decimal places, signified by zeros, and keep adding these zeros until the remainder is zero. We get:

$$16 \quad 11.{}_{5}\overset{14}{0}{}_{6}\overset{12}{0}{}_{5}\overset{8}{0}{}_{3}\overset{0}{0}$$
$$\quad\quad 0.\ 6\ 8\ 7\ 5$$

Since $\frac{11}{16} = 0.6875$, then:

$$28\frac{11}{16} = 28.6875$$

Method 2. We multiply the decimals. The method of multiplying decimals is to ignore all decimal points and multiply as usual:

$$\begin{array}{r} 675 \\ \times\ 425 \\ \hline {}^{4}2\,{}^{6}4\,{}^{4}0\,{}^{2}4\,5\,5 \end{array} \quad \text{Product: } 286{,}875$$

We note that our product is too large. What must we do to correct this? What does it mean to ignore, or remove, the decimal points?

We changed 4.25 to 425. What did we do?

$$4.25 \times 100 = 4.25 \times 10^2 = 425$$

We then changed 6.75 to 675. Again, we multiplied:

$$6.75 \times 100 = 6.75 \times 10^2 = 675$$

Notice what happens to the decimal point in 4.25 and 6.75 respectively. We multiplied both numbers by the second power of 10 so, in 425 and 675, the decimal point has been shifted two places to the right:

4.25 (decimal point between 4 and 2) to 425 (assumed decimal point is after 5)
6.75 (decimal point between 6 and 7) to 675 (assumed decimal point is after 5)

Since we multiplied both factors by 100, the product must be 100 × 100 = 10,000 ($10^2 \times 10^2 = 10^4$) times too large. To fix this error, we must divide 286,875 by 10,000. Since 10,000 = 10^4 and we are dividing by a power of 10, all we need to do is shift the decimal point in 286,875 four places to the left:[7]

286,875 (assumed decimal point is after 5) to 28.6875 (decimal point between the first 8 and 6)

Our adjustment generates a product that agrees with our first method demonstrating that the two methods are dancing in sync:

$$28\frac{11}{16} = 28.6875 \text{ yd}^2$$

If we are not dealing with actual measurements, 28.6875 is the correct answer. Since 4.25 and 6.75 are measurements, we look for the least precise measurement except with a slight twist due to the higher order nature of multiplication.[8] The product must contain the same number of significant figures as the factors with the fewest significant figures. If one of the factors is not a measurement, e.g., you want to multiply a measurement by 4, this factor has an unlimited number of significant figures.

Since both factors have three significant figures, the product, rounded so, is:

$$28.7 \text{ yd}^2$$

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[7] We will revisit powers of 10 and division of decimals in Lesson 7.6.

[8] Since multiplication is one order higher than addition, we must consider the possible propagation of measurement error by the operation of multiplication. Therefore, we consider least significant digits instead of least precise measurement.

> When multiplying measurements, the product must contain the same number of significant figures as the factors with the fewest significant figures. If one of the factors is not a measurement, e.g., you want to multiply a measurement by 4, this factor has an unlimited number of significant figures.

Let's work through two more examples:

Example 2. What is the product of 2.5 and 3.42?

Estimate: $3 \times 3 = 9$

First, we remove the decimal points and multiply:

$$\begin{array}{r} 342 \\ \times\ 25 \\ \hline \overset{2}{6}\overset{2}{3}\overset{1}{4}0 \end{array} \quad \text{Product: } 8550$$

This product is $10 \times 100 = 1000$ ($10^1 \times 10^2 = 10^3$) times too large. We shift the decimal point in 8550 three places to the left:

8550 (assumed decimal point is after 0) to 8.550 (decimal point between 8 and the first 5)

If we are dealing with measurements, there are two significant digits in 2.5 and three significant digits in 3.42. Our answer should only have two significant figures, the less precise measurement. 8.550, rounded to two significant figures, is:

8.6

Example 3. What is the product of 3.10 and 4.520?

Estimate: $3 \times 5 = 15$

Remove the decimal points and ending zeros and multiply:

$$\begin{array}{r} 452 \\ \times\ 31 \\ \hline 1\overset{1}{2}\overset{1}{9}12 \end{array} \quad \text{Product: } 14{,}012$$

This product is $10 \times 100 = 1000$ ($10^1 \times 10^2 = 10^3$) times too large. We shift the decimal point in 14,012 three places to the left. We get:

14.012

If we are dealing with measurements, there are three significant digits in 3.10 and four significant digits in 4.520. Our answer should only have three significant figures, the less precise measurement. 14.012, rounded to three significant figures is:

14.0

Remember, 14 and 14.0 are not the same in terms of measurements since 14.0 is ten times as precise as 14.

THE PRINCIPLE OF MULTIPLYING DECIMALS

In summary, when multiplying decimals:

Step 1. Remove all decimal points from the two factors to generate two integers. Removing the decimal points means you are multiplying both numbers by a power of 10 and that power will be the number of digits to the right of the decimal point before it is removed.
Step 2. Multiply the two integers.
Step 3. Return the decimal point to its proper place in the product by shifting to the left as many places as there are decimal places in both factors. Returning the decimal point is equivalent to dividing by a power of 10.
Step 4. Adjust the product to consider significant figures if the factors are measurements.

Always remember what you are doing when you remove decimal points in the factors, i.e., you are multiplying by a power of 10. And, when you return the decimal point in the product by shifting the decimal point to the left, you are dividing by a power of 10.

EXERCISES

Define the following symbol/terms:

1. ±
2. Meniscus
3. Calibration
4. Density

Field Project:

5. (a) Measure the volume of three different amounts of liquids using the beaker, graduated cylinder, and buret. (b) Explain the reading error for each measurement.

Answer the following questions mentally, stating the answer as a decimal:

6. What is:
   (a) 8 times 1/10?
   (b) 7 times 0.4?
   (c) 3 times 0.7?
7. What is:
   (a) $\frac{3}{10} \cdot 3$?
   (b) 7 × 0.3?
   (c) 8 × 0.6?
   (d) 0.2 × 16?
8. What is:
   (a) $\frac{3}{100} \cdot 3$?
   (b) 8 × 0.02?
   (c) 0.06 × 3?
   (d) 5 × 0.03?

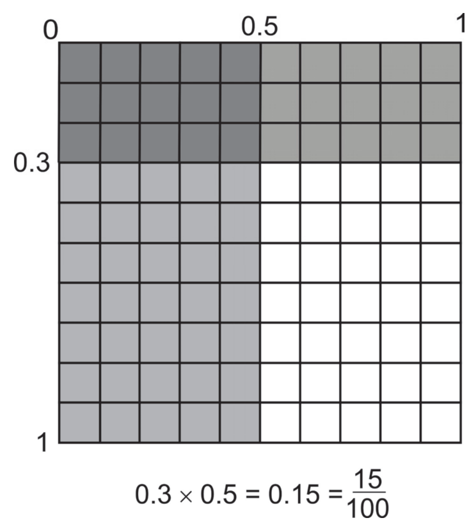

Figure 5. Multiplying decimals and area.

9. What is:
   (a) $\frac{7}{10} \cdot \frac{8}{10}$?
   (b) 0.2 × 0.4?
   (c) 0.5 × 0.5?
   (d) 0.3 × 0.7?
10. What is:
   (a) $\frac{7}{100} \cdot \frac{7}{10}$?
   (b) 0.3 × 0.02?
   (c) 0.11 × 0.8?
   (d) 0.5 × 0.15?
11. What is:
   (a) $\frac{8}{100} \cdot \frac{3}{100}$?
   (b) 0.02 × 0.05?
   (c) 0.12 × 0.08?
12. Determine:
   (a) (5)(0.7)
   (b) (5)(0.07)
   (c) (5)(0.007)
   (d) (5)(0.0007)

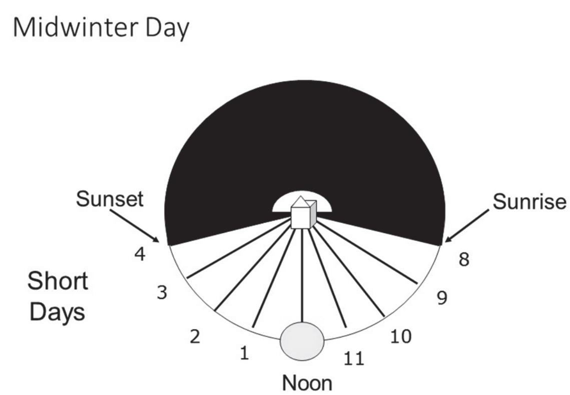

Figure 6. Midwinter in the Northern Hemisphere.

Multiply and state the product in decimals: (Assume that these factors are not measurements.)
13. 0.72 by 0.9
14. 819.2 by 0.819
15. 112.91 by 0.325
16. 8.125 by 0.0875
17. 0.623 by 100
18. 0.623 by 1000

19. 0.0101 by 75
20. 53,000 by 53 thousandths
21. 41 tenths by 1021 hundredths

Round the rectangular areas with the following dimensions using significant digits:
22. 0.72 cm by 0.9 cm
23. 819.2 mm by 0.819 mm
24. 112.91 m by 0.325 m
25. 8.125 km by 0.0875 km

Answer the following questions:
26. Explain the difference between accuracy and precision.
27. If a digital weight scale reads 235.8 pounds, what are the boundary conditions of this measurement?
28. What is the value of 50 sets of encyclopedias at $97.50 a set?
29. What is the price of 12 cords of wood at $125.75 a cord?
30. What is the value of 150.5 acres of land at $43,000 an acre?
31. What is the cost of 5.125 kilograms of silver at $1.25 a gram? (Note: 1 kg = 1000 g)
32. A grocer sold 50 cartons of powdered milk at $6.50 a carton and 25.3 grams of cheese at $0.63 a gram to Mr. Gilliam. Mr. Gilliam paid for these goods with $500 in cash. Calculate how much change to Mr. Gilliam receives. (Note: round to the nearest cent.)

Calculate x in the following equations: (Consider significant figures for measurements. Remember: do all operations within parentheses first. Then do exponentiation first, multiplication and division second and addition and subtraction last.)
33. $(1.309 \text{ cm})(0.051 \text{ cm})(0.5 \text{ cm}) = x$
34. $\$818.50 - \$24.50 \cdot 4 + \$169.99 = x$
35. $(\$1000 - \$475.25)(0.25 + 2.75) = x$
36. $5.015 \times 0.625 + 21.5 \times 10^3 = x$

Figure 7. American coins (late 19th and early 20th century). Source: United States Postal Service.