DIVISIBILITY *RULES*

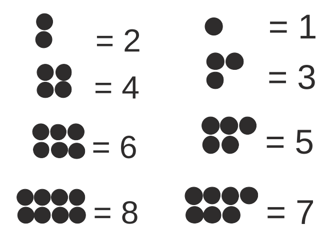
BY JAMES D. NICKEL

Up to this point in our study, we have explored addition, subtraction, and multiplication of whole numbers. Recall that, on average, out of every 100 arithmetical problems you encounter 70 will require addition, 20 will require multiplication, 5 will require subtraction, and 5 will require division. Although least used, division is a part of everyday life. Parents, in providing inheritances for their children, generally seek to divide the estate equally between their children. When four people play with a deck of 52 cards, they seek to divide the cards equally to each player.

We are now ready to explore the nature of division and, to speak truthfully, division is one of the hardest processes for a student to master *because it is the most complicated of all the arithmetical operations.* Note what was said about it in 1600:

> Division is esteemed one of the busiest operations of Arithmetick, and such as requireth a mynde not wandering, or setled uppon other matters.[1]

So, we shall tread this division ground lightly before we start digging. We shall provide a few lessons of "interlude" first that I think you will find very enjoyable.



The ancient Greeks of the Classical era (ca. 600–300 BC) were fascinated with the structure of number, especially the natural numbers. For the philosophers of this period, the study of the structure of natural numbers was a favorite "thought" hobby. We have already seen one example of this in the classification of even and odd numbers. To review, they learned a great detail about the nature of numbers by arranging the pebbles in a variety of representations. Arranging pebbles in rectangles or squares using only 2 rows was their way of defining even numbers. They also noted that they could not arrange the other numbers in like fashion (one pebble was always "left over" or "too short"). These non square and non rectangular arrangements defined odd numbers.

By limiting the arrangement of even numbers to square or rectangles with two rows only introduces us to the concept of divisibility. Divisibility means the capacity of a group of objects to be divided *evenly.*

What does this mean? To help us understand this concept, we must consider what division means and it has two different meanings: (1) sharing and (2) containment.

In both types of division, a given group of objects (called a set) is divided into *equally sized sets.* Sharing and containment answer two different questions. In sharing division, the question is, "How many does each set share?" In containment division, the question is, "How many sets are there?"

Let's provide some examples. Sharing division is what we are used to in everyday life. I have $8 and I want to share this money equally amongst my two children. *How much money does each child receive?* I must divide (or distribute) the $8 into two equal sets (the two children). Each child should receive $4.

In containment division, the roles of the number of sets and the number of items *are reversed.* I have $8 and each child gets $4. *How many children are there?* There are 2 children. The same concept can be expressed as how many times does $4 go into $8? Or, how many times does 8 contain 4?

It is time for some definitions. The general form of an operation of division is:

In Latin (15^{th} century),	In English,
numerus dividendus ÷ numerus divisor = numerus querendus	dividend ÷ divisor = quotient

Division is the process (1) of finding *how many times* one number is contained in another of the same kind or (2) of finding *one of the equal parts* of a number. The *dividend* is the number to be divided. The *divisor* is the number by which to divide. The *quotient* is the result of the division and it shows *how many times* the dividend

[1] Cited in Jan Gullberg, *Mathematics: From the Birth of Numbers* (New York: W. W. Norton, 1997), p. 120.

contains the divisor. The *sign of division* of ÷ and it is read "divided by." It shows that the number before it is to be divided by the one after it. Thus 8 ÷ 4 is read "eight divided by 4." We can also write a division problem with two numbers separated by a bar (called the *fraction bar*): $\frac{8}{4}$ and 8 ÷ 4 have the same meaning (note: we can also write $\frac{8}{4}$ and 8/4). Notice the dot above and the dot below the sign of division represent the dividend and divisor respectively. Thus, the sign of division is a perfect symbolic representation.

When you can divide a set exactly into equal parts (with no remainder or anything left over) then you have divisibility. For example if I wanted to divide $9 (and I just had $1 bills) equally among two children, I could not do it. Each would get $4, and $1 would be left over (I could cut it in half but it would be worthless or I could get change for the dollar in coins and try to divide them equally). The part of the dividend remaining when the division is not *exact* is called the *remainder*. A little thought would lead to this observation: *the remainder must always be less than the divisor*.

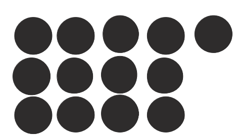
In finding *how many times* one number is contained in another (containment division), the divisor and the dividend are like numbers and the quotient is an abstract number. In finding one of the *equal parts* of a number (sharing division), the dividend and quotient are like numbers and the divisor is an abstract number.

Since division is the inverse of multiplication, we can check a division problem using multiplication. The dividend will be equal to the product of the divisor and the quotient, plus the remainder. Now we can define *divisibility* with more accuracy. A number is said to be divisible by another number when, after dividing, the remainder is zero.

Look again at the ancient Greek definition of even and odd numbers. For pebble representations, why the limitation of *two* rows? The Greeks were thinking in terms of divisibility. An *even number* is divisible by 2. The two rows demonstrate this divisibility principle. An *odd number* is *not* divisible by 2. Why? There is always one pebble *left over as a remainder*. Note that 1 is an odd number because 1 ÷ 2 = 0 with a remainder of 1. This means that you cannot arrange 1 pebble into two rows (quotient = 0). You just have 1 pebble (remainder of 1).

Divisibility is a key property of whole numbers. Understanding divisibility will help you master division. In conclusion, the ancient Greek pebble representation provides us with a universal principle: *every even number is divisible by 2*.

If two rows of pebbles can establish divisibility by 2, then how would you establish divisibility by 3? By 4? By 5? By 6? By 7? By 8? I think you know the process. To check divisibility by 3, you see if the pebbles represent a number can be arranged as a square or rectangle limited to 3 rows. Use pennies and see if you can arrange the numbers 6, 9, 12, 15, and 18 as either perfect rectangles or squares. Then, try arranging 13 in rows of 3. How many are left over? There is a remainder of 1. Hence, 13 ÷ 3 = 4 R 1. We indicate the remainder, if any, using the capital letter R followed by the number that comprises the remainder. The quotient of 3 means you will have 3 rows, 4 columns, and 1 left over. All division problems are founded upon this "pebble picture."



Is there a general rule that we can use to determine if a number is divisible by 3? First, list the multiples of 3 from 3 to 54 and let's see if we can find a pattern. In the first column, list the multiples. In the second column, I want to sum the digits until you have a one digit sum. For example, the sum of the digits of 99 is 9 + 9 = 18. 18 is a two digit sum so 1 + 8 = 9 (a one digit sum). Is 9 divisible by 3? Yes. Hence, I make that claim that 99 is divisible by 3. Complete your table and then compare with the one below.

Multiples of 3	*Sum of the digits*
3	3

6	6
9	9
12	3
15	6
18	9
21	3
24	6
27	9
30	3
33	6
36	9
39	3
42	6
45	9
48	3
51	6
54	9

The pattern is that the sum of the digits form a repeating sequence of 3, 6, 9. Since we know that 3, 6, and 9 are divisible by 3, then we can reasonably conclude that if the sum of the digits of any number is divisible by 3, then that number is divisible by 3! Let's investigate why this is true. Let's consider the number 354. The sum of its digits is 3 (3 + 5 + 4 = 12; 1 + 2 = 3). Our rule claims that 354 is divisible by 3. Let's investigate 354. According to place value, 354 = 300 + 50 + 4.

Follow this reasoning carefully as we look at each of these addends. Consider 300. Is it divisible by 3? Since 3 × 100 = 3, then 300 is divisible by 3. Next, consider 50. Is it divisible by 3? We know that 48 and 51 are divisible by 3. Therefore, 50 has a remainder of 2 when divided by 3. Finally, consider 4. It has a remainder of 1 when divided by 3. 300 has remainder 0 when divided by 3, 5 has remainder 2, and 4 has remainder 1. Let's add the remainders: 0 + 2 + 1 = 3 and 3 is divisible by 3. Our leftovers (remainders) form a group that is divisible by 3. Hence, 354 is divisible by 3! Let's try one more: Is 673 divisible by 3? Inspect the table:

Addends	*Divisible by 3?*	*Remainder*
600	Yes	0
70	No	1
3	Yes	0
Sum of digits = 7		

Sum of remainders is 1. Hence, 673 is *not* divisible by 3. Note that 6 + 7 + 3 = 16; 1 + 6 = 7 and 7 has a remainder of 1 when divided by 3.

Is 1506 divisible by 3?

Addends	*Divisible by 3?*	*Remainder*
1000	No	1
500	No	2
6	Yes	0
Sum of digits = 3		

The sum of remainders is 3 (which is divisible by 3). Hence, 1506 is divisible by 3 (Remainder = 0). Note that 1 + 5 + 6 = 12; 1 + 2 = 3 and 3 is divisible by 3 (Remainder = 0). How do we know that 1000 ÷

3 has a remainder of 1? Since we know the 3, 6, and 9 are divisible by 3, then we can conclude that 300, 600, and 900 are divisible by 3. If 3 is divisible by 3, then 100 times 3 is divisible by 3. In fact, *any natural number* times 3 is a multiple of 3 and all multiples of 3 are divisible by 3! We also know that 4, 7, and 10 have remainder 1 when divided by 3. Likewise, 400, 700, and 1000 have remainders of 1 when divided by 3. Also, 5, 8, and 11 have remainder of 2 when divided by 3. Hence, 500, 800, and 1100 have remainder of 2 when divided by 2.

Is 35,782 divisible by 3?

Addends	*Divisible by 3?*	*Remainder*
30,000	Yes	0
5000	No	2
700	No	1
80	No	2
2	No	2
Sum of digits = 7		

The sum of the remainders is 2 + 1 + 2 + 2 = 7 and 7 is not divisible by 3 (Remainder = 1). Hence, 35,782 is not divisible by 3. Note that 3 + 5 + 7 + 8 + 2 = 25; 2 + 5 = 7 (Remainder = 1).

Are you seeing why all we have to do is sum the digits in any number and then test that sum to see if it is divisible by 3? If the sum is divisible by 3, then that number is divisible by 3. The decimal system and place value requires this conclusion.

Let's try one more. Is 3,467,276 divisible by 3?

Addends	*Divisible by 3?*	*Remainder*
3,000,000	Yes	0
400,000	No	1
60,000	Yes	0
7000	No	1
200	No	2
70	No	1
6	Yes	0
Sum of digits = 8		5

The sum of the remainders is 5. The sum of the digits is 8. Both 5 and 8 have remainder *2* when divided by 3. Hence, 3,467,276 is *not* divisible by 3.

We now know two rules (actually 4) of divisibility.

1. *A number is divisible by 2* if it is even *or if its ones digit is divisible by 2* (why is this true?).
2. *A number is divisible by 3* if the sum of its digits is divisible by 3.
3. *A number is divisible by 5* if its ones digit is 0 or 5 (from our analysis of the laws of multiplication).
4. *A number is divisible by 10* if its ones digit is 0 (from our analysis of the laws of multiplication).

Let's conclude this lesson by determining the rule for divisibility by 6. First, note that 6 is divisible by 3. Hence, our rule for divisibility by 3 (summing digits) will have a part to play in developing a rule for divisibility by 6. As before, let's generate a table of multiples of 6 from 6 to 108 and notice the pattern(s) revealed.

Multiples of 6	*Sum of the digits*
6	6
12	3
18	9
24	6

DIVISIBILITY *RULES*

BY JAMES D. NICKEL

Multiples of 6	*Sum of the digits*
30	3
36	9
42	6
48	3
54	9
60	6
66	3
72	9
78	6
84	3
90	9
96	6
102	3
108	9

The pattern is that the sum of the digits for a repeating sequence of 6, 3, 9. Since we know that 3, 6, and 9 are divisible by 3, then we can invoke the same rule as the rule for divisibility by 6? Think carefully before you answer. Look at the multiples of 6 and compare them with the multiples of 3. What do you notice? Note that multiples of 3 contain both odd an even numbers. Multiples of 6 are *even numbers only*. We have our rule! A number is *divisible by 6* if its ones digit is even *and* the sum of its digits is divisible by 3. Let's try a few.

> *A Law of Logic:*
> A proposition is a statement that is either true or false. A logical conjunction is an operation on two propositions normally joined by the conjunction "***and***." For a logical conjunction to be true, then *both* propositions must be true. For example, for a number to be divisible by 6 it must be even (proposition 1) ***and*** it must also be divisible by 3 (proposition 2).

Is 567 divisible by 6? It is odd, so we know immediately that 567 is not divisible by 6. No summing of digits is necessary to determine this. Just look at the last digit.

Is 7842 divisible by 6? It is even so let's sum the digits (remember to make use of recording tens!): 7 + 8 + 4 + 2 = 21; 2 + 1 = 3. Hence, 7842 is divisible by 6. Too easy!

We are collecting tools to help us with division. We have 5 in our bag so far. Let's next explore the rules for 4, 8, and 12.

Let's review our divisibility rule toolkit:

1. *A number is divisible by 2* if it is even or if its ones digit is divisible by 2.
2. *A number is divisible by 3* if the sum of its digits is divisible by 3.
3. *A number is divisible by 5* if its ones digit is 0 or 5.
4. *A number is divisible by 10* if its ones digit is 0.
5. *A number is divisible by 6* if its ones digit is even *and* the sum of its digits is divisible by 3. Or, if a number is divisible by both 2 and by 3, then it is divisible by 6. Why?

Let's investigate divisibility by 4. We begin by inspecting its multiples in order to determine a pattern.

Multiples of 4	
4	52
8	56
12	60
16	64
20	68
24	72
28	76
32	80
36	84
40	88
44	92
48	96
	100

We have not summed the digits for a reason. The first observation to make is that 100 is a multiple of 4. This means that 200, 300, 400 and *all multiples of 100 are also multiples of 4*. This is an important observation because when we consider a number like 458,248, we know that 400,000 is divisible by 4 (since it is a multiple of 4), 50,000 is divisible by 4 (since it is a multiple of 4), 8000 is divisible by 4 (same reason), and 200 is a multiple of 4. That leaves us, if we divide 458,248 by 100, with a remainder of 48 (the last two digits). So, all we have to do is inspect the last two digits of any sized number to determine if that number is divisible by 4. In our example, 48 is divisible by 4. Hence, 458,248 is divisible by 4! Again, too easy!

So, to determine if any number is divisible by 4 all we have to know is if the last two digits in that number are divisible by 4. In our table, we have all the ammunition that we need: the multiples of 4 from 4 to 96. Instead of memorizing these 24 numbers, let's see if we can make things a little easier yet. The first two numbers, 4 and 8, should be easy to remember. The numbers 4008 and 6304 are divisible by 4 (the last two digits, 08 and 04 are divisible by 4). Let's now focus on the 22 two-digit numbers. Let's add these digits with a slight twist. Let's add the ones digit to twice the tens digit. Inspect this table.

Multiples of 4	*Ones digit plus twice the tens digit*
12	2 + 2×1 = 4
16	6 + 2×1 = 8
20	0 + 2×2 = 4
24	4 + 2×2 = 8
28	8 + 2×2 = 12
32	2 + 2×3 = 8
36	6 + 2×3 = 12
40	0 + 2×4 = 8
44	4 + 2×4 = 12
48	8 + 2×4 = 16
52	2 + 2×5 = 12
56	6 + 2×5 = 16

Multiples of 4	*Ones digit plus twice the tens digit*
60	0 + 2×6 = 12
64	4 + 2×6 = 16
68	8 + 2×6 = 20
72	2 + 2×7 = 16
76	6 + 2×7 = 20
80	0 + 2×8 = 16
84	4 + 2×8 = 20
88	8 + 2×8 = 24
92	2 + 2×9 = 20
96	6 + 2×9 = 24

By this method, we have reduced the total list to remember to 4, 8, 12, 16, 20, and 24 (6 numbers instead of 22). Also, these 6 numbers are all divisible by 4. Note that this "twist" works for 4 and 8 also. Considering 4 and 8 as two digits, we get 04 and 08 (from our work in multiplication, we are used to this). When we multiply the tens digit (which is 0) by 2, we get 0. Adding the ones digit, we get 4 and 8 (both are divisible by 4).

So, here is our rule, stated in two ways: A *number is divisible by 4* if its last two digits are divisible by 4 or if its ones digit plus twice its tens digit is divisible by 4.

Let us now consider the rule for divisibility by 8. We should have noticed that since 2 × 3 = 6, then if a number is divisible by 2 *and* by 3, then it will be divisible by 6. Can we use the same reasoning with 8? Can we say, if a number is divisible by both 2 and by 4 (2 × 4 = 8), then that number is divisible by 8? Take the number 12, 20 or 28. All three numbers are all divisible by 4 *and* by 2, yet not by 8. Hence, we cannot apply the same reasoning.[2]

Just as we noted that all multiples of 100 are also multiples of 4, we can conclude that all multiples of 1000 are multiples of 8 (since 8 × 125 = 1000). Hence, 525,000, 1,342,000, etc. are all multiples of 8. To determine if *any* number is divisible by 8, then we must only inspect the last three digits, the hundreds place, the tens place, and the ones place.

First let's note all the two-digit numbers divisible by 8. We should memorize these numbers.

Multiples of 8	
8	56
16	64
24	72
32	80
40	88
48	96

To help us develop a rule for larger numbers, we will extend our rule for divisibility by 4. Inspect this table containing the first 20 three-digit multiples of 8:

[2] Technically, we cannot apply the reasoning because the relationship of 2 to 3 is different than the relationship of 2 to 4. Can you see what is different? Hint: 2 × 2 = 4 but no whole number times 2 equals 3 or 4 is a multiple of 2, but 3 is not a multiple of 2.

Multiples of 8	*Ones digit plus 2 times the tens digit plus 4 times the hundreds digit*
104	4 + 2×0 + 4×1 = 8
112	2 + 2×1 + 4×1 = 8
120	0 + 2×2 + 4×1 = 8
128	8 + 2×2 + 4×1 = 16
136	6 + 2×3 + 4×1 = 16
144	4 + 2×4 + 4×1 = 16
152	2 + 2×5 + 4×1 = 16
160	0 + 2×6 + 4×1 = 16
168	8 + 2×6 + 4×1 = 24
176	6 + 2×7 + 4×1 = 24
184	4 + 2×8 + 4×1 = 24
192	2 + 2×9 + 4×1 = 24
200	0 + 2×0 + 4×2 = 8
208	8 + 2×0 + 4×2 = 16
216	6 + 2×1 + 4×2 = 16
224	4 + 2×2 + 4×2 = 16
232	2 + 2×3 + 4×2 = 16
240	0 + 2×4 + 4×2 = 16
248	8 + 2×4 + 4×2 = 24
256	6 + 2×5 + 4×2 = 24

From this table, we see a pattern. A **number is divisible by 8** if its ones digit plus twice its tens digit plus 4 times its hundreds digit is divisible by 8. Note that this rule also works for two digit numbers. This rule is another way of saying that a number is number by 8 if its last three digits are divisible by 8.

Is 488 divisible by 8?

8 + 16 + 16 = 40. 40 is divisible by 8. Hence, 488 is divisible by 8

Is 714 divisible by 8?

4 + 2 + 16 = 22. 22 is *not* divisible by 8. Hence, 714 is *not* divisible by 8.

Let's extend our reasoning to determine divisibility by 12. We know that 12 = 3 × 4. We already know the rules for 3 and 4 and since 4 is not a multiple of 3, so a *number is divisible by 12* if it is divisible by *both* 3 *and* 4.

Is 384 divisible by 12? The sum of its digits is 6 and 84 is divisible by 4. Yes, 384 is divisible by 12.

Is 5424 divisible by 12? The sum of its digits is 6 and 24 is divisible by 4. Yes!

Is 107,582 divisible by 12? The sum of its digits is 5. Since this number does *not* meet the divisibility by 3 test, then one of the conditions for divisibility by 12 fails. If one condition fails in a "both/and" statement, then we need not go any farther. 107,582 is *not* divisible by 12. You could also observe that 82 is not divisible 4 (another failure).

Our toolkit for divisibility now includes 8 rules:

1. Divisibility by 2.
2. Divisibility by 3.
3. Divisibility by 4.

4. Divisibility by 5.
5. Divisibility by 6.
6. Divisibility by 8.
7. Divisibility by 10.
8. Divisibility by 12.

What about the rules for divisibility by 7, 9, and 11 (the numbers remaining)? Officially, there are very complex tests for divisibility by 7 and 11. There is a relatively easy test for divisibility by 7 (although it does take some effort). Divisibility by 11 requires some pattern recognition. We will not investigate these rules in this essay. Divisibility by 9 is fascinating. Let's now look into the wonders of 9.

To aid us in conquering division, we are learning a good number of divisibility rules. In the arithmetic of the decimal system, 9 *is very interesting number*. When the decimal system was first employed (5th to 9th century AD), mathematicians recognized the fascinating properties of 9 and developed a time-honored rule of "casting out nines."[3] Leonardo of Pisa (also known as Fibonacci) introduced this rule to Medieval Europe through his book *Liber abaci* (1202) as a check for arithmetic. Remember, Fibonacci was trying to popularize the Hindu-Arabic system of decimal numbers and this rule probably finds its origin in either the Hindu or Arabic cultures.

Before we explore what this means, let's investigate. Ready? Let's first list the multiples of 9 from 9 to 99.

9
18
27
36
45
54
63
72
81
90
99

Using this table, let's consider division by 9 for the multiples of 10 from 10 to 80.

Division problem	*Quotient*	*Remainder*
10/9	1	1
20/9	2	2
30/9	3	3
40/9	4	4
50/9	5	5
60/9	6	6
70/9	7	7
80/9	8	8

What is the pattern? When dividing the tens by 9, the remainder equals the number of tens in the dividend. Let's continue by considering division by 9 for multiples of 100 from 100 to 800.

Division problem	*Quotient*	*Remainder*
100/9	11	1

[3] The principles revealed in this lesson can be universally applied to any base. In base 6, the principle would be called "casting out fives." In base 12, "casting out elevens." In base 60, the principle would be called "casting out 59s."

Division problem	*Quotient*	*Remainder*
200/9	22	2
300/9	33	3
400/9	44	4
500/9	55	5
600/9	66	6
700/9	77	7
800/9	88	8

How did we get the quotients without actually dividing (something we have not explained yet)? We followed the pattern revealed in the first table. 90/9 = 10 (with remainder of 0) since 90 is a multiple of 9. Therefore, following the pattern, 100/9 = 11, 200/9 = 22, etc. We next use multiplication to find the remainders: 11 × 9 = 99 and 100 – 99 = 1 (the remainder), 22 × 9 = 198 and 200 – 198 = 2 (the remainder), etc.

Note that the same pattern is discernible when we divided hundreds by 9 as when we divide tens by 9. Do you think the same pattern will be noticeable when we divide thousands by 9? *Yes!*

How can we now use this pattern to check to see if a number, any number, is divisible by 9? Is 1,341 divisible by 9?

Using place value, we expand 1,341 = 1,000 + 300 + 40 + 1. Next, look at the table.

Divide by 9	*Remainder*
1000/9	1
300/9	3
40/9	4
1/9	1
Sum of remainders =	9

The sum of the remainders is 9. Since this sum is divisible by 9, then 1,341 is divisible by 9! Here is the rule: *any number is divisible by 9 if the sum of its digits is divisible by 9.*

Let's try the rule out on a few numbers:

Number	*Sum of digits*	*Sum of the sum of the digits*	*Divisible by 9?*
278	17	8	No
918	18	9	Yes
39,438	27	9	Yes
152,112	12	3	No
6,789,425	41	5	No
19,836,378	45	9	Yes

Note that we added another column entitled the "Sum of the sum of the digits." Similar to the divisibility by 3 rule, if the sum of digits is two digits (or more), then we can sum those digits repeatedly until we get a sum that is 1 digit. If this number is 9, the original number is divisible by 9. Note also: any time we see a 9 in our digit sums, we can *"cast it out."*

Number	*Casting out nines*	*Sum of the digits (and sum of the sum of the digits)*	*Divisible by 9?*
~~27~~8	2 + 7	8	No
~~918~~	9 and 1 + 8	0	Yes
3~~9~~,438	9	9	Yes
1~~52~~,11~~2~~	5 + 2 + 2	3	No

DIVISIBILITY *RULES*

BY JAMES D. NICKEL

Number	*Casting out nines*	*Sum of the digits (and sum of the sum of the digits)*	*Divisible by 9?*
6,~~789~~,~~425~~	9 and 7 + 2 and 4 + 5	5	No
~~19~~,~~836~~,378	9 and 1 + 8 and 3 + 6	9	Yes

If our sum is 9 or 0, then the number is divisible by 9. Why 0? With 918, the sum of its digits is 9 + 1 + 8 = 18; 1 + 8 = 9. So, if we have a number like 918 where we can "cast out nines" (the 9 and the 1 + 8), our sum becomes 0. Whenever we see this, *we should recognize that the sum of the digits is really divisible by* 9.

Why can we cast out nines like this? Consider 278 = 200 + 70 + 8. 200/9 has remainder of 2. 70/9 has remainder of 7. 2 + 7 = 9 (divisible by 9). Hence, we can "cast out" the 2 and the 7 (which sum to 9). We, therefore only consider 8 which, being less than 9, is not divisible by 9.

There is an additional benefit to "casting out nines." We can find the remainder of any number divided by 9. Consider 8,216,458. What is the remainder, if any, when this number is divided by 9? We Cast out 8 + 1 and 4 + 5. We show this "casting out" as follows: ~~8~~,2~~1~~6,~~45~~8. The digits we have left are 2, 6, and 8. We add these digits: 2 + 6 + 8 = 16; 1 + 6 = 7. Hence, 8,216,458 is not divisible by 9 and its remainder, when divided by 9, is 7.

What is the remainder, if any, when 54,878 is divided by 9? We cast out the 5 and 4. What is left is 8, 7, and 8. 8 + 7 + 8 = 23; 2 + 3 = 5. The remainder is 5!

Before the advent of calculators and computers, people (primarily accountants) used the "sum the digit's procedure" to check the accuracy of an addition problem. Here is how it works. Add the following numbers:[4]

	9	2	7
		9	4
	3	0	8
	1	1	8
+			4
1	3	2	1
	1	3	
1	4	5	1

Now, add the digits in the sum. Cast out 4 + 5. We have 1 + 1 = **2**. To check, add the digits in each of the addends and then sum this total (summing the digits on the total):

				Process	*Sum of digits*
	9	2	7	Cast out 9 and 2 + 7	0
		9	4	Cast out 9	4
	3	0	8	11; 1 + 1	2
	1	1	8	Cast out 1 + 8	1
+			4		4
					11; 1 + 1 = **2**

Note that the sum of the digits in the sum and the sum of the digits in the addends are equal (numbers in **bold**). *This procedure does not guarantee that your answer is always correct*, but when these two numbers are unequal, you know for certain that you have made a calculation error.

Check the following sum by casting out nines:

						Process	*Sum of digits*
	1	2	9	0	5	Cast out 9; 1 + 2 + 5	8

[4] For the reader not familiar with adding numbers "left to right" using complements, see Edward Stoddard, *Speed Mathematics Simplified* (New York: Dover Publications, [1962, 1965] 1994.

	2	8	0	7	2	Cast out 7 + 2; 2 + 8 = 10	1
		9	0	0	1	Cast out 9	1
		6	2	1	4	Cast out 6 + 2 + 1	4
+		7	7	2	8	Cast out 7 + 2; 7 + 8 = 15	6
	3	2	8	0	0		**2**
	3	1	1	2			
	6	3	9	2	1	Cast out 9 and 6 + 3; 2 + 1	**3**

Since 2 ≠ 3, then we have made a mistake somewhere. Do you know where it is? The mistake is in the ones position. It should be 0 instead of 1. If we make this correction, then our sums match; i.e., 2 = 2.

Let's try another. Make sure that you understand the "Process" column.

							Process	*Sum of digits*
			4	1	2	5	Cast out 4 + 5; 1 + 2	3
	2	3	4	6	1	6	Cast out 2 + 3 + 4; 6 + 1 + 6 = 13	4
		9	1	0	0	9	Cast out the two 9s; 1	1
				4	5	6	Cast out 4 + 5; 6	6
		2	8	6	7	8	2 + 8 + 6 + 7 + 8 = 31	4
+	1	6	5	4	2	5	Cast out 5 + 4 and 1 + 6 + 2; 5	5
	3	0	2	1	7	9		**5**
	2	2	2	1	3			
	5	2	4	3	0	9	Cast out 9 and 5 + 4; 2 + 3	**5**

Our answer checks!

The same "casting out of nines" works for checking multiplication problems, too. Multiply the following.

	3	7	5		
×		2	6		
	0	7	5	0	
		1	2	5	0
		9	7	5	0

Sum the digits in the product: Casting out the 9, we get 7 + 5 = 12 = **3**. Sum the digits in each factor: 375 = 15 = 6 and 26 = 8. Now *multiply* 6 × 8 = 48. Summing the digits in 48, we get 12 = **3**. Tentative accuracy has been confirmed!

Check the following products by casting out nines. Write your work in your notebook and check with the solutions below.

1250
× 146
182,500

236,189
× 27,458
6,486,277,562

Problem 1:
Multiplicand: 1 + 2 + 5 = 8.
Multiplier: 1 + 4 + 6 = 11; 2.
8 × 2 =16; **7**.

Product: Cast out 1 + 8, **7**. The answer is correct.

Problem 2:
Multiplicand: 2 (cast out 3 + 6, 1 + 8, and 9)
Multiplier: 8 (cast out 2 + 7 and 4 + 5).
8 × 2 =16; **7**.
Product: Cast out 1 + 8, **7**. The answer is correct

This now completes our toolkit for division (except for 7 and 11). In conclusion, let's review what we have learned.

1. *A number is divisible by 2* if it is even *or* if its ones digit is divisible by 2.
2. *A number is divisible by 3* if the sum *or* its digits is divisible by 3.
3. *A number is divisible by 4* if its last two digits are divisible by 4 *or* if its ones digit plus twice its tens digit is divisible by 4.
4. *A number is divisible by 5* if its ones digit is 0 *or* 5.
5. *A number is divisible by 6* if its ones digit is even *and* the sum of its digits is divisible by 3 *or* if a number is divisible by *both* 2 *and* by 3, then it is divisible by 6.
6. *A number is divisible by 8* if its ones digit plus twice its tens digit plus 4 times its hundreds digit is divisible by 8.
7. *A number is divisible by 9* if the sum of its digits is divisible by 9.
8. *A number is divisible by 10* if its ones digit is 0.
9. *A number is divisible by 12* if it is divisible by 4 *and* by 3.