# Finding the Derivative of the Sine Function

By James D. Nickel

Given $y = \sin\theta$ (where $\theta$ is in radians), what would the derivative be? Given the unit circle (radius = 1) with angle $\theta$ (theta), then, by definition:

$PA = \sin\theta$
$OA = \cos\theta$.

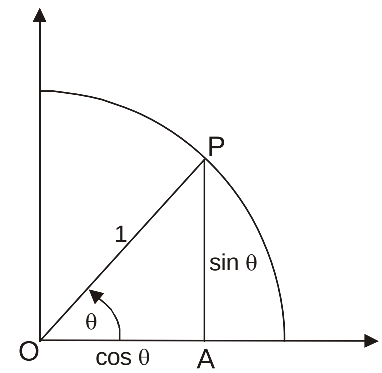


Figure 1

Note that as $\theta$ changes (from 0 to $\frac{\pi}{2}$; remember $\frac{\pi}{2}$ radians equals 90°), $\sin\theta$ also changes (from 0 to 1). In that same interval (0 to $\frac{\pi}{2}$), $\cos\theta$ changes from 1 to 0. This fact would indicate that the cosine function and the sine function are inverses of each other.

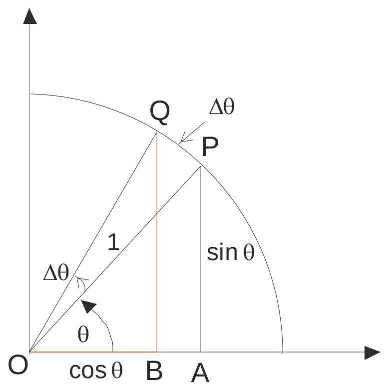


Figure 2

How does $\sin\theta$ vary as $\theta$ varies? To find out, we must calculate the derivative of $y = \sin\theta$. Using the method of increments, we add $\Delta\theta$ to $\theta$ and see (Figure 2) what happens with $y + \Delta y = \sin(\theta + \Delta\theta)$.

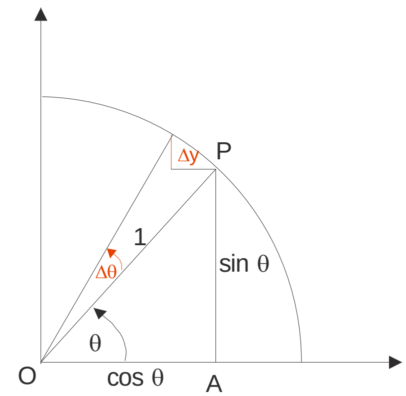


Figure 3

In Figure 3, we have increased $\theta$ by $\Delta\theta$. By doing so PA has *increased* to QB where $QB = \sin(\theta + \Delta\theta)$. At the same take OA has *decreased* to OB where $OB = \cos(\theta + \Delta\theta)$. $\Delta\theta$ also represents the radian measure of the arc from P to Q. Also remember that $\Delta\theta$ is just a "little bit" or infinitesimally small.

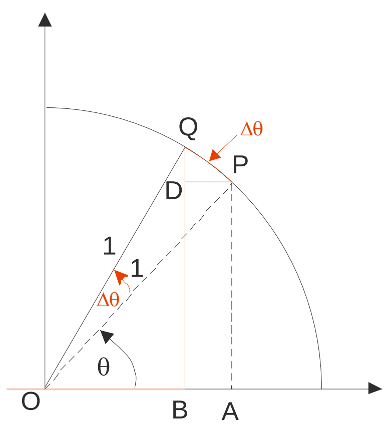


Figure 4

Our goal is to find $QD = \Delta y$ (Figure 4). To do this, we must show $\Delta QDP \sim \Delta PAO$. We then let $\Delta\theta \to 0$. Hence, $\frac{\Delta y}{\Delta\theta} = \frac{DQ}{\Delta\theta} = \frac{OA}{PO}$. From this, we can determine the derivative of the function $y = \sin\theta$.

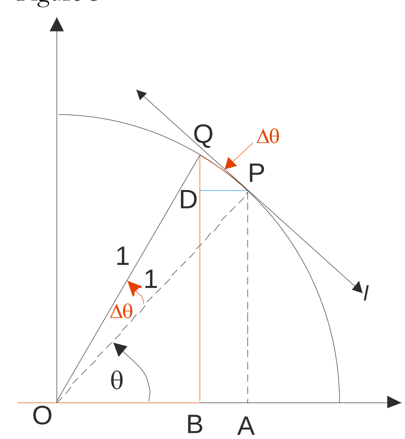


Figure 5

Let's now draw the tangent line *l* to the circle at point P. By one of Euclid's propositions (Book III, Proposition 18[1]), we know that $l \perp \overline{PO}$ (remember, the symbol ⊥ means "perpendicular to"). That means that $\angle QPO = 90°$.

In Figure 5, we construct $\overline{DP}$ such that it is parallel to $\overline{OA}$ (in symbols, $\overline{DP} \parallel \overline{OA}$). Another one of Euclid's propositions states that if two parallel lines are cut by transversal[2] ($\overline{OP}$ in this instance), then alternate

[1] T. L. Heath, *Euclid: The Thirteen Books of The Elements* (New York: Dover Publications, 1956), 2:44-45. This proposition states that the tangent line intersects the circle's radius at a right angle.

interior angles are equal (Book I, Proposition 29[3]). In this case, $\theta = \angle DPO$. Since $\angle QPO$ is a right angle (i.e., $\angle QPO = 90°$), then $\angle DPO$ and $\angle DPQ$ are complementary; i.e., $\angle DPQ = 90° - \theta$.

Since $\angle DPQ = 90° - \theta$, then $\angle DQP = \theta$. We have now have two triangles, $\Delta QDP$ and $\Delta PAO$ that are similar to each other (in symbols, $\Delta QDP \sim \Delta PAO$) because both are right triangles and $\angle DQP = \angle POA = \theta$.

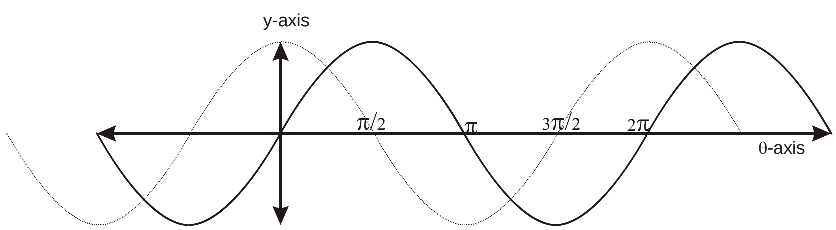
Now consider DQ. What does it represent? It represents the change in y (i.e., $\Delta y$) resulting from the change in $\theta$ (i.e., $\Delta\theta$). In symbols:

$$\frac{\Delta y}{\Delta\theta} = \frac{DQ}{\Delta\theta} = \frac{\sin(\theta + \Delta\theta) - \sin\theta}{\Delta\theta}$$

Remember, $\frac{\Delta y}{\Delta\theta}$ represents the infinitesimal rate of change of y divided by $\theta$. This is our derivative formula. In $\Delta QDP$, $\cos\theta = \frac{DQ}{\Delta\theta}$ (side adjacent over the hypotenuse). As $\Delta\theta$ gets infinitesimally small, QP (the hypotenuse) converges to $\Delta\theta$. Because $\Delta QDP \sim \Delta PAO$, then $\frac{DQ}{\Delta\theta} = \frac{OA}{PO}$. Since PO = 1 and OA = $\cos\theta$, then:

$$\frac{\Delta y}{\Delta\theta} = \frac{DQ}{\Delta\theta} = \frac{\sin(\theta + \Delta\theta) - \sin\theta}{\Delta\theta} = \frac{OA}{PO} = \frac{\cos\theta}{1} = \cos\theta$$

As we let $\Delta\theta$ approach 0 as a limit, then $y' \sin\theta) = \cos\theta$. This means as $\theta$ increases, $\sin\theta$ increases at a instantaneous rate of $\cos\theta$. We can visualize this with the following graph:




y = sin θ

The solid line graph represents $y = \sin\theta$ (the sine curve). The dotted line graph represents $y' = \cos\theta$ (the cosine curve). Note that when $\theta = 0$, then $\cos\theta = 1$. This means that the slope of the line tangent to the sine curve at 0 is 1 (Do you see this?). When $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$. This means that the slope of the line tangent to the sine curve at $\frac{\pi}{2}$ is 0 (the slope is parallel to the $\theta$-axis). Hence, the cosine curve traces the derivative of the sine curve at every point on the sine curve.

By similar reasoning, we can calculate the derivative of $\cos\theta$. If $y = \cos\theta$, then $y' = -\sin\theta$.

Congratulations on following this classic mathematical argument. This logic is an example of the *crème de la crème*[4] of deductive analysis.

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[2] Transverse is Latin for "to turn across." In mathematics, a transversal is a line or line segment that intersects a system of other lines (in our example, a system of parallel lines).

[3] Heath, 1:311-314.

[4] *crème de la crème* is French for "superlative."