Finding the Derivative of e^x

By James D. Nickel

The number *e* (approximately equal to 2.7182818) is a fascinating number.[1] Let's consider taking the derivative of $y = e^x$. To do this, we first note that e^x is defined as follows[2]:

$$y = e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

The Swiss mathematician Leonhard Euler (1707-1783) developed a power series[3] for e^x:

$$y = e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

Here we have, in essence, a series of polynomial functions. Taking the derivative of $y = e^x$ is as easy as taking the derivative of each term of this polynomial (this is called the *sum rule for differentiation*).

To do this we must remember that, given $y = x^n$, $y' = nx^{n-1}$. This gives us the rule for taking the derivative of each term except the first. Before we start computing, note that for each term except the first, the denominator is a constant. For example, we can rewrite $x^2/2!$ as:

$$\frac{1}{2}x^2$$

In general, if $y = ax^n$, we can show that $y' = nax^{n-1}$. Therefore, we can calculate the derivative for this term as follows:

$$y = \frac{1}{2}x^2 \Rightarrow y' = 2\left(\frac{1}{2}\right)x = x$$

This takes care of the third term. The derivative of the first term, 1, being a constant, is 0. The derivative of the second term, x, is 1.

The derivative of the fourth term is calculated as follows:

$$y = \frac{1}{3\cdot 2\cdot 1}x^3 \Rightarrow y' = 3\left(\frac{1}{3\cdot 2\cdot 1}\right)x^2 = \frac{1}{2}x^2$$

The derivative of the fifth term follows the same procedure:

$$y = \frac{1}{4\cdot 3\cdot 2\cdot 1}x^4 \Rightarrow y' = 4\left(\frac{1}{4\cdot 3\cdot 2\cdot 1}\right)x^3 = \frac{1}{3!}x^3$$

Here is what we get after we sum all these derivatives:

$$y = e^x \Rightarrow y' = 0+1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots = e^x$$

This is astonishing. The derivative of e^x is e^x! e^x is the only function in all of mathematics in which its derivative equals itself. This means that the instantaneous rate at which e^x changes is e^x. The central role of this function in mathematics and science is a direct consequence of this fact. One can find many phenomena in God's world in which the rate of change of some quantity is proportional to the quantity itself.

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[1] For an instructive study of *e*, see Eli Maor, *e: The Story of a Number* (Princeton: Princeton University Press, 1994).
[2] See www.biblicalchristianworldview.net/Mathematical-Circles/forgetMultiplicationTable.pdf
[3] See www.biblicalchristianworldview.net/Mathematical-Circles/eulerCrownJewel.pdf

We find this happening with the rate of decay of a radioactive substance. When a hot object is put into a cooler environment (e.g., hot cocoa on ice), the object cools at a rate proportional to the difference in temperatures (called Newton's law of cooling). When sound waves travel through any medium (air or otherwise), their intensity decreases in proportion with the distance from the source of the sound. We have seen how money compounded continuously (at every instant) grows proportionally with time. The growth of a population (given certain requirements and limitations) grows proportionately with time.