THE DANCE OF THE BINOMIAL THEOREM

BY JAMES D. NICKEL

The binomial theorem is an important mathematical proposition, especially in Calculus. It relates to the expansion of the power of a binomial, meaning "two term," expression of the form:

$$(a+b)^n$$

By multiplying, we can determine the algebraic expression of this binomial for values of n from 1 to 5. We write:

$(a+b)^1 = a+b$ (2 terms)

$(a+b)^2 = a^2+2ab+b^2$ (3 terms)

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$ (4 terms)

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$ (5 terms)

$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$ (6 terms)

The coefficients of the terms interpenetrate with Pascal's Triangle (Figure 1):

$(a+b)^0$ (1 term)

Note: If we let $k = a + b$, then $k^0 = 1$.

$(a+b)^1$ (2 terms)

$(a+b)^2$ (3 terms)

$(a+b)^3$ (4 terms)

$(a+b)^4$ (5 terms)

$(a+b)^5$ (6 terms)

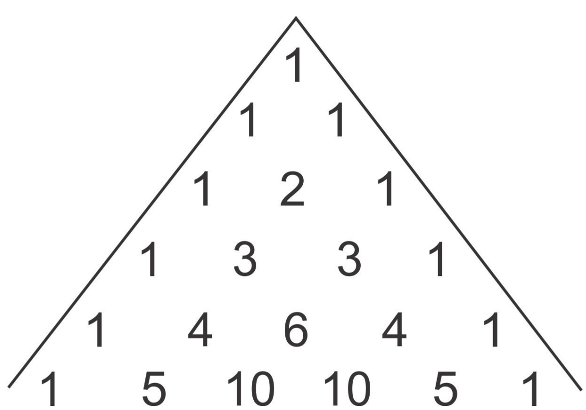

Figure 1

We note that each number in the triangle greater than 1 is the sum of the numbers immediately to its left and to its right in the line above.

To calculate the numbers 2, 3, 4, 6, and 10, we do this:

2 = 1 + 1
3 = 1 + 2 or 3 = 2 + 1
4 = 1 + 3 or 4 + 3 + 1
6 = 3 + 3
10 = 4 + 6 or 10 = 6 + 4

If we let t represent the number of the term of the expansion with exponent n, and we let $r = t - 1$, then we can represent the coefficient of each term $a^{n-r}b^r$ as:

$$\frac{n!}{(n-r)!r!}$$

This is the familiar combination formula ${}_nC_r$ for counting objects. We can, therefore, redraw Pascal's Triangle (Figure 2).

$$ {}_0C_0 $$
$$ {}_1C_0 \quad {}_1C_1 $$
$$ {}_2C_0 \quad {}_2C_1 \quad {}_2C_2 $$
$$ {}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3 $$
$$ {}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4 $$
$$ {}_5C_0 \quad {}_5C_1 \quad {}_5C_2 \quad {}_5C_3 \quad {}_5C_4 \quad {}_5C_5 $$

Figure 2

To find the coefficient of the third term of the expansion of $(a+b)^5$ we do this:

We know $n = 5$ and $t = 3$. Therefore, $r = t - 1$ or $r = 3 - 1 = 2$. We also conclude that $n - r = 5 - 2 = 3$.
From the combination formula, we write:

$$\frac{5!}{(5-2)!2!}=\frac{5!}{3!2!}=\frac{5\cdot \overset{2}{\cancel{4}}\cdot \overset{1}{\cancel{3\cdot 2\cdot 1}}}{\underset{1}{\cancel{3\cdot 2\cdot 1}}\cdot \underset{1}{\cancel{2}}\cdot 1}=10$$

We can now state the **Binomial Theorem**: For $n \in \mathbb{N}$:

$$(a+b)^n = a^n + \frac{n!}{(n-1)!1!}a^{n-1}b + \frac{n!}{(n-2)!2!}a^{n-2}b^2 + \frac{n!}{(n-3)!3!}a^{n-3}b^3 + \ldots + \frac{n!}{1!(n-1)!}ab^{n-1} + b^n$$

where the coefficient of $a^{n-r}b^r$ is $\frac{n!}{(n-r)!r!}$.

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To prove this theorem, we invoke the dance of Mathematical Induction.

Step 1. We first let $n = 1$. We note:

$$(a+b)^1 = a^1 + b^1 = a + b$$

We have proved the first condition of induction.

Step 2. Next, we assume the formula is true for $n = k$. We write:

$$(a+b)^k = a^k + \frac{k!}{(k-1)!1!}a^{k-1}b + \frac{k!}{(k-2)!2!}a^{k-2}b^2 + \frac{k!}{(k-3)!3!}a^{k-3}b^3 + \ldots + \frac{k!}{1!(k-1)!}ab^{k-1} + b^k$$

To show that this formula is true for $n = k + 1$, we multiple each term on the left and right of the equal sign by $(a + b)$. We write: (Make sure you confirm the algebraic work.)

$$\begin{aligned}(a+b)^{k+1} = (a^{k+1} + a^k b) &+ \frac{k!}{(k-1)!1!}(a^k b + a^{k-1}b^2) \\ &+ \frac{k!}{(k-2)!2!}(a^{k-1}b^2 + a^{k-2}b^3) \\ &+ \frac{k!}{(k-3)!3!}(a^{k-2}b^3 + a^{k-3}b^4) \\ &+ \ldots + \frac{k!}{1!(k-1)!}(a^2b^{k-1} + ab^k) \\ &+ ab^k + b^{k+1}\end{aligned}$$

We group like variable factors for the expression on the right of the equal sign: (Again, make sure you confirm the algebraic work.)

$$\begin{aligned}(a+b)^{k+1} = a^{k+1} &+ \left[1 + \frac{k!}{(k-1)!1!}\right]a^k b \\ &+ \left[\frac{k!}{(k-1)!1!} + \frac{k!}{(k-2)!2!}\right]a^{k-1}b^2 \\ &+ \left[\frac{k!}{(k-2)!2!} + \frac{k!}{(k-3)!3!}\right]a^{k-2}b^3 \\ &+ \left[\frac{k!}{(k-3)!3!} + \frac{k!}{(k-4)!4!}\right]a^{k-3}b^4 \\ &+ \ldots + \left[\frac{k!}{1!(k-1)!} + 1\right]ab^k \\ &+ b^{k+1}\end{aligned}$$

We observe that the coefficient of $a^{(k+1)-r}b^r$ is:

$$\frac{k!}{(r-1)!(k-r+1)!} + \frac{k!}{r!(k-r)!} = \frac{k!r + k!(k-r+1)}{r!(k-r+1)!} = \frac{k!(k+1)}{(k+1-r)!r!} = \frac{(k+1)!}{(k+1-r)!r!}$$

[Note: What must you do to get from $\frac{k!}{(r-1)!(k-r+1)!}+\frac{k!}{r!(k-r)!}$ to $\frac{k!r+k!(k-r+1)}{r!(k-r+1)!}$?]

We have established, therefore, that the formula is true for $n = k + 1$; i.e., the coefficient $\frac{(k+1)!}{(k+1-r)!r!}$ is the coefficient $\frac{n!}{(n-r)!r!}$ when $n = k + 1$. Since the truth of the assertion for any natural number k implies its truth for $k + 1$, we have proved the second condition of induction.

Thus, by the principle of the dance of Mathematical Induction, the Binomial Theorem is true for $n \in \mathbb{N}$.

QED