ARITHMETIC SEQUENCES

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A number sequence is a list of numbers (each number in the sequence is called a *term*) that follows a pattern or rule. From the beginning of recorded history (from the cities of Sumer, the location of present day Iraq), we find references to number sequences in farming and astronomical contexts. Arithmetic sequences play an important role in many fascinating areas of modern day mathematics.

The sequence we are most familiar with is the counting numbers or {1, 2, 3, 4, …}.[1] The pattern or rule is easy to discern (add 1 to the current number to get the next number). In an *arithmetic sequence*, you calculate each successive term by adding the *same* number (this number may be positive or negative). Thus, there is a *common difference* between any two adjacent terms in the sequence. Another word for common difference is "rate of change." In counting number sequence, this rate of change is illustrated in this table:

Source: iStockPhoto

Term	*Number*	*Rate of Change (Common Difference)*
1st	1	-
2nd	2	1
3rd	3	1
4th	4	1
etc.	etc.	etc.

We understand "rate of change" in the counting number sequence by saying, "As the term increases by 1, the number increases by 1."

In track and field, the runners in the 400m hurdles race must leap over a succession of evenly spaced hurdles (a wooden framed barrier) therefore forming an arithmetic sequence. The first hurdle is 45m from start. The second hurdle is 80m from start. The third is 115m and the fourth hurdle is 150m. Since the common difference, or rate of change between hurdles, is 35m, the remaining hurdles will be at the following distances from the start:

Fifth: 185m
Sixth: 220m
Seventh: 255m
Eighth: 290m
Ninth: 325m
Tenth: 360m

The table below illustrates the hurdles sequence:

Hurdle	*Distance from Start*	*Rate of Change (Common Difference)*
1st	45m	-
2nd	80m	35m
3rd	115m	35m
4th	150m	35m
5th	185m	35m
6th	220m	35m
7th	255m	35m
8th	290m	35m
9th	325m	35m

[1] Remember, the three dots (…) are an *ellipsis* meaning to continue with the same pattern *ad infinitum.*

Hurdle	*Distance from Start*	*Rate of Change (Common Difference)*
10^{th}	360m	35m

We understand this "rate of change" by saying, "As the hurdle number increases by 1, the hurdle distance increases by 35m."[2]

Here are more examples of arithmetic sequences:

1, 9, 17, 25, 33, 41, … (common difference is 8)
1, 3, 5, 7, 9, 11, … (common difference is 2; this is the sequence of odd numbers)
10, 20, 30, 40, … (common difference is 10)
22, 11, 0, -11, … (common difference is -11)

Is this sequence arithmetic?

10, 11, 14, 19, 26, …

Since the difference between terms is 1, 3, 5, 7, …, this sequence is *not* arithmetic. It follows a neat pattern, though!

The analysis of arithmetic sequences teaches us how to "put on our mathematical thinking caps." For example, let's say that we know the first and fourth terms of an arithmetic sequence, 12 and 57 respectively. What are the second and third terms?

If we let x and y stand for the second and third terms, the sequence goes like this: 12, x, y, 57. What can we do? First, we note the difference between the fourth and first term. It is 57 – 12 = 45. Second, we note that here are three segments:

First segment: From 12 to x.
Second segment: From x to y.
Third segment: From y to 57.

If we divide 45 by 3, we will find the common difference: 45 ÷ 3 = 15. Therefore, x = 27 and y = 42. The sequence looks like this: 12, 27, 42, 57, …

We have been looking at arithmetic sequences that do not end (indicated by the ellipsis …). We call these lists *infinite* arithmetic sequences. We can stop the sequence list at any time, stand back, and ask some questions. One important question would be, "Can we find an easy way to find the *sum* of a finite arithmetic sequence?"

Look at an example: 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 (the common difference is 2). Let S be the sum of this finite arithmetic sequence. We can find S by adding each term (S = 144), but is there a shorter way? Finding a shorting way means finding a *formula*. The study of mathematics reveals a host of formulas, short cut ways to solve hard problems. Formulas are revelatory of the power of mathematics.

Let's use our brains to find a formula for S. First, we ask, "How is each term calculated?" We let t_n (read "t sub n"[3]) stand for any term of the sequence. Hence, t_1 (read "t sub 1") is the first term, t_2 (read "t sub 2") is the second term, etc. Here is how we calculate each term of this sequence:

$t_1 = 1$
$t_2 = 3 = 1 + 2$
$t_3 = 5 = 3 + 2 = 1 + 2 + 2$
$t_4 = 7 = 5 + 2 = 1 + 2 + 2 + 2$

[2] This common difference, or rate of change, is also called *slope*. Slope comes into play in the analysis of the graphs of linear equations and, most importantly, in the analysis of the "action of a curved line" at a specific point in the Calculus.
[3] In the symbol t_n, *n* is called a *subscript*.

$t_5 = 9 = 7 + 2 = 1 + 2 + 2 + 2 + 2$
etc.

Let's see if we can find some sort of pattern. For the first term, t_1, we add 2 zero times. For the second term, t_2, we add 2 one time. For t_3 we add 2 two times. For t_4, we add 2 three times. We can write this pattern like this:

$t_1 = 1 + 0 \times 2$
$t_2 = 1 + 1 \times 2$
$t_3 = 1 + 2 \times 2$
$t_4 = 1 + 3 \times 2$
$t_5 = 1 + 4 \times 2$
etc.

A pattern develops from the study of these particulars. We can generate a universal, a *formula*, to calculate the value of any term in this sequence. For any n^{th} term:

$$t_n = 1 + (n - 1) \times 2$$

or

$$t_n = 2(n - 1) + 1$$

(more algebraically presentable)

This formula does not help us find S, but it is an exercise in mathematical analysis. Look at the sequence again: 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23. Note that the first and last term (1 and 23) sum to 24 (1 + 23) and the same goes for the second and next to last term, etc. Hence, we can find S this way:

1 + 23 = 24
3 + 21 = 24
5 + 19 = 24
7 + 17 = 24
9 + 15 = 24
11 + 13 = 24

Since we are summing 24 a total of six times, S = 6 × 24 = 144! Still no formula, though. To help us reach our goal, we can write the sequence two ways (first term to last term and last term to first term):

1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23
23 + 21 + 19 + 17 + 15 + 13 + 11 + 9 + 7 + 5 + 3 + 1

Add the columns: 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 = 12 × 24 = 288. Since we wrote the sequence twice, we divide 288 by 2 to get S = $\frac{288}{2} = 144$.

What did we do? First, we summed the first and last term (= 24). Second, we multiplied this sum by the number of terms (24 × 12 = 288). Third, we divided 288 by 2 (S = $\frac{288}{2} = 144$).

Are you ready for a genuine mathematical analysis of what we did? We start with a general arithmetic sequence with its sum S:

Equation 1: $S = t_1 + t_2 + t_3 + \cdots + t_{n-2} + t_{n-1} + t_n$

Second, we reverse the terms:

Equation 2: $S = t_n + t_{n-1} + t_{n-2} + \cdots + t_3 + t_2 + t_1$

Third, we add Equation 1 to Equation 2:

Equation 3: $2S = (t_1 + t_n) + (t_2 + t_{n-1}) + (t_3 + t_{n-2}) + \ldots + (t_3 + t_{n-2}) + (t_2 + t_{n-1}) + (t_1 + t_n)$

We note on the right side of Equation 3, each term is equal to the same number; i.e., $(t_1 + t_n) = (t_2 + t_{n-1}) = (t_3 + t_{n-2}) = \ldots = (t_3 + t_{n-2}) = (t_2 + t_{n-1}) = (t_1 + t_n)$. Let $t_1 + t_n$ be this number and we note that $t_1 + t_n$ is added *n* times. We rewrite Equation 3:

Equation 4: $2S = n(t_1 + t_n)$

We divide both sides of Equation 4 by 2:

Equation 5: $S = \frac{n(t_1 + t_n)}{2}$. Our general formula!

Equation 5 gives you a symbolic representation of *the* formula to find the sum of *any* arithmetic sequence. We have found our *universal* that enables us to sum *any particular* finite arithmetic sequence. We need to know only three facts:

1. The number of terms, n.
2. The first term, t_1.
3. The last term, t_n.

Mathematicians have a more compact way to express the general arithmetic sequence $t_1 + t_2 + t_3 + \ldots + t_n$. They use $\sum$, capital Greek letter sigma, and they let it stand for *sum.* $\sum_{i=1}^{n} t_n$ means "sum the first *n* terms of an arithmetic sequence that starts with the term t_1 and ends with the term t_n." The letter "i" is called the *index* of the summation notation. Using "sigma" notation, $t_1 + t_2 + t_3 + \ldots + t_n = \sum_{i=1}^{n} t_n$. If we let $S = \sum_{i=1}^{n} t_n$, then $S = \sum_{i=1}^{n} t_n = \frac{n(t_1 + t_n)}{2}$. Symbols shorten the number of characters or numbers you must write and mathematicians do not like exerting any extra effort!

We apply this formula to our example: 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23. First, we must make sure that the sequence is arithmetic (i.e., 2 is the common difference separating each term). Second, we count the number of terms: n = 12. Third, we identify the first term: $t_1 = 1$. Fourth, we identify the last term: $t_n = 24$. Fifth, we apply the formula:

$$S = \sum_{i=1}^{n} t_n = \frac{n(t_1 + t_n)}{2} = \frac{\overset{6}{\cancel{12}}(1 + 23)}{\underset{1}{\cancel{2}}} = 6 \times 24 = 6(20 + 4) = 120 + 24 = 144$$

In conclusion, let's use our newly found formula to calculate the number of presents given during the classic "The Twelve Days of Christmas."

Day 1: 1 partridge in a pear tree
Day 2: 2 turtle doves
Day 3: 3 French hens
Day 4: 4 calling birds
Day 5: 5 golden rings
Day 6: 6 geese a-laying
Day 7: 7 swans a-swimming
Day 8: 8 maids a-milking
Day 9: 9 ladies dancing
Day 10: 10 lords a-leaping
Day 11: 11 pipers piping
Day 12: 12 drummers drumming

This is an arithmetic sequence (common difference = 1). What is the sum of 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12? We note that n = 12, t_1 = 1 and t_n = 12. Applying the formula, we get:

$$S = \sum_{i=1}^{n} t_n = \frac{n(t_1 + t_n)}{2} = \frac{\overset{6}{\cancel{12}}(1+12)}{\underset{1}{\cancel{2}}} = 6 \times 13 = 6(10+3) = 60 + 18 = 78$$

A goodly number of Christmas presents, indeed!