Archimedes: From the Law of the Lever to Formulas for a Sphere

by James D. Nickel

For the philosopher Plato (427-347 BC), there was a general scorn of the material realm. For him, the philosophic spirit is one that transcends the material and gazes upon the perfections of the immaterial ideal. Since Greek Geometry was a study of ideal forms (the "perfect" circle, triangle, square, etc.), it was the ideal road leading to intellectual perfection.[1] Euclid's (ca. 300 BC) *Elements*, one of the most famous mathematics textbooks of all time, followed this Platonic scheme by building a system of geometry starting with foundational axioms/definitions that lead to a host of geometric theorems via deductive analysis. Commenting on Euclid's work, mathematics historian Herbert Meschkowski (1909-1990) said, "Every effort was made not to rely on the unsupported intuition but to construct geometry as a scientific system with a precise axiomatic foundation."[2] In Euclid's treatise, there is not a single instance of the application of geometry to the physical world or of any heuristic starting point for any of his theorems.

A few generations after the initial publication of *Elements,* the technician Archimedes of Syracuse (ca. 287-212 BC) introduced heuristics into geometric analysis. He still retained much of Plato in his thinking. According to 2nd century Roman historian Plutarch, in *Life of Marcellus* (chapter 14), Archimedes considered "mechanical work and every art concerned with the necessities of life an ignoble and inferior form of labor and therefore exerted his best efforts only in seeking knowledge of those things in which the good and the beautiful were not mixed with the necessary."[3] Forced by the pressing circumstances of his times (war, economics, agriculture, and thievery to name a few), his skill at "ignoble" work was astonishing.[4]

Unlike Euclid, Archimedes used reasoning about physical processes (mechanics) to help him solve problems in mathematics. He said:

> Certain theorems first became clear to me by means of a mechanical method. Then, however, they had to be proved geometrically since the method provided no real proof. It is obviously easier to find a proof when we have already learned something about the question by means of the method than it is to find one without such advance knowledge. That is why, for example, we must give Democritus, who was the first to state the theorems that the cone

[1] The Greeks perfected the method of abstraction, the method of laying aside unimportant details so that the essence of an issue can be considered. Abstraction is both a gift of God and essential to thinking. Unfortunately, the Greeks absolutized abstraction and generally disregarding empirical analysis (the foundation of operational science). For the "ivory towered" Platonist, the intellect is the primary and only way to properly perceive the good, the beautiful and the true. Historically, embracing the Biblical view that God is the creator of the human mind that can think abstractly and the physical world that reflects abstract and principled patterns formed the womb for the viable birth of operational science, a birth that inert Greek thinking could never generate.

[2] Herbert Meschkowski, *Ways of Thought of Great Mathematicians: An Approach to the History of Mathematics* (San Francisco: Holden-Day, 1964), p. 14.

[3] Cited in G. E. R. Lloyd, *Greek Science After Aristotle* (New York: W. W. Horton, 1973), pp. 93-94. Lloyd's source is the Loeb translation of B. Perrin, *Plutarch's Lives*, Vol. 5 (Cambridge: Harvard University Press, 1917). Plutarch lived from ca. 46 to 120 AD.

[4] Archimedes discovered many truths about mechanics and hydrostatics.

is a third of the cylinder and the pyramid of the prism, but who did not prove them, as much credit as we give to Eudoxus, who was the first to prove them.[5]

We shall proceed to investigate how Archimedes used his method to develop the formula for the volume and surface area of a sphere.[6]

To one's wonderment, Archimedes' heuristic starting point for deriving the sphere formulas was the lever. As a child, I played with my friends on a seesaw with a goal to "balance" it. Depending on our weights, we would change positions (move closer to or farther from the fulcrum[7]) so that both of us would "even the saw" by stopping its motion when it was parallel to the ground. In the terminology of physics, a seesaw balances when the *moments* on each side of its fulcrum are equal.[8] Figure 1 illustrates the Archimedean *Law of the Lever* where w_1 and w_2 represent the respective weights of two objects and d_1 and d_2 represent the distance from the center of gravity of these weights to the fulcrum of the lever. The lever balances when the moments are equal; i.e., $w_1d_1 = w_2d_2$.

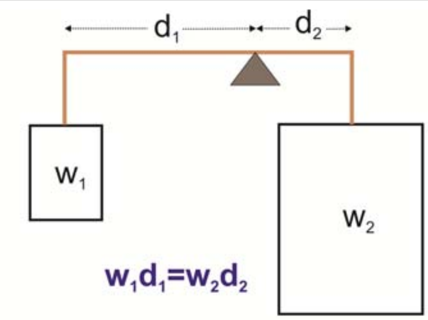

Figure 1

The principle of the lever was known to the Greeks long before the birth of Archimedes. *It is what Archimedes did with this principle that is unprecedented.* Archimedes reasoned from this law to develop important formulas in solid geometry (or stereometry). Instead of balancing weights about a fulcrum, Archimedes envisioned balancing geometric objects about their center of gravity. He considered the two-dimensional situation in Figure 2, balancing a square with a circle. The radius of the circle is *r* and its distance from the fulcrum (about its center, the center of the circle's gravity) is x. Considering area analogous with weight, Archimedes calculated the moment of the circle to the fulcrum as $\pi r^2 x$. He then let r = distance of the square from the fulcrum. Hence, by the *Law of the Lever*, A_s (area of the square) must be:

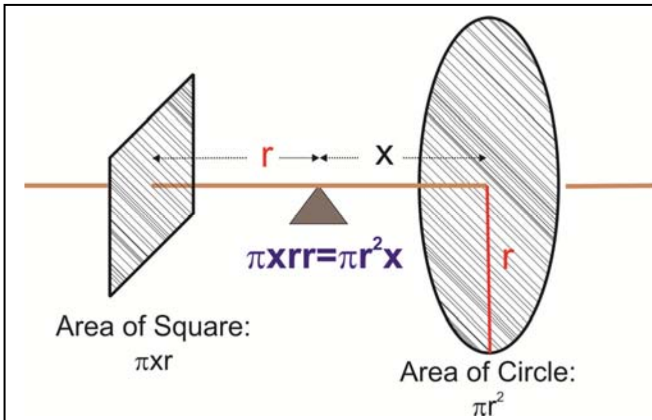

Figure 2

$$A_s r = \pi r^2 x \Leftrightarrow A_s = \frac{\pi r^2 x}{r} = \pi r x$$

[5] *The Method of Archimedes*. In T. L. Heath, ed. *The Works of Archimedes* (New York: Dover Publications, [1912] 2002), p. 13.
[6] I will follow a summary of the method of Archimedes by William M. Priestley, *Calculus: A Liberal Art* (New York: Springer-Verlag, [1974] 1988), pp. 347-350.
[7] Fulcrum is Latin for "support." In the context of a lever (of which a seesaw is an example), the fulcrum is the support, or point of rest, on which a lever turns in moving a body.
[8] A *moment* is defined as the product of a weight and its distance from the fulcrum of a lever.

Next, Archimedes considered Figure 3 where the area of the square, πrx, is replaced by the combined area of the two circles. The center of gravity of the two circles on the left is the same as the center of gravity of the square in Figure 2. We let x = the radius of the bottom circle on the left. Hence, the area of the bottom circle is πx^2. To maintain equilibrium, the area of the top square must be $\pi rx - \pi x^2$.

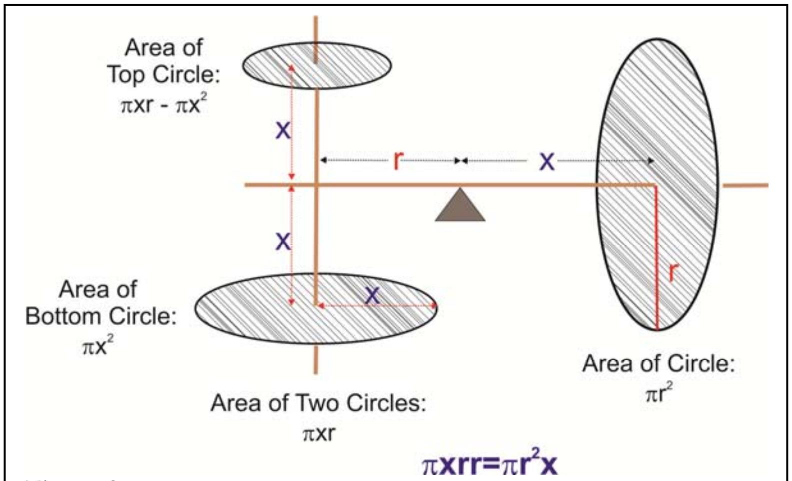

Figure 3

Now, Archimedes considered generating a sphere around the top circle, a cone around the bottom circle, and a cylinder around the right circle such that each vertical slice through the cylinder is exactly balanced by a corresponding pair of horizontal slices in the sphere and the cone (see Figure 4).[9]

Archimedes was now in a position to develop a formula for the volume of the sphere. As he noted, the "laughing philosopher" Democritus (ca. 460-ca. 370 BC) had already shown the relationship between the volume of a cone and that of a cylinder of equal base and height; and similarly for the pyramid and prism; i.e., the volume of the cone is $\frac{1}{3}$ the volume of the cylinder and the volume of a pyramid is also $\frac{1}{3}$ the volume of a prism.

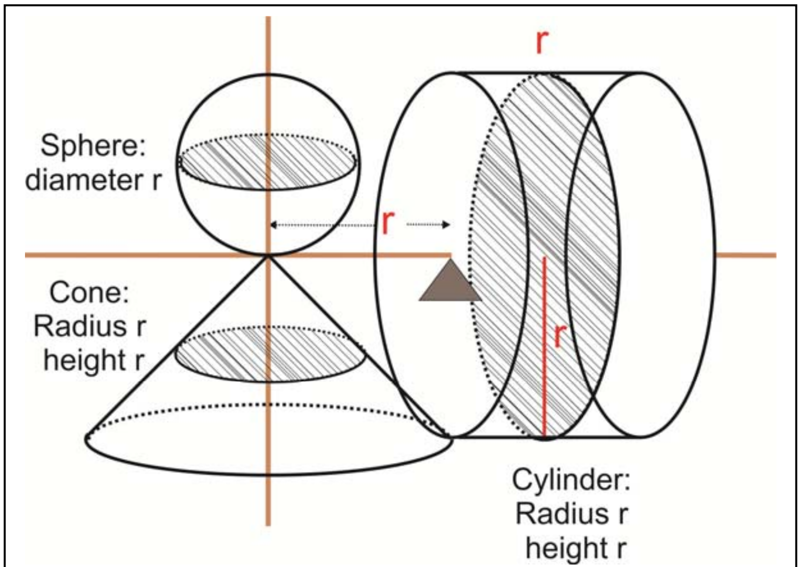

Figure 4

Since the configuration in Figure 4 balances, we can apply the *Law of the Lever*. Let V_{cone} = volume of the cone, V_{sphere} = volume of the sphere, and $V_{cylinder}$ = volume of the cylinder. We get:

$$(V_{cone} + V_{sphere})r = (V_{cylinder})\frac{r}{2}$$

Note: $\frac{r}{2}$ is the length of the moment arm of the center of gravity of the cylinder. Dividing this equation by r, we get:

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[9] This method was repeated by the Italian mathematician Bonaventura Cavalieri (1598-1647) and it portends the methods of the integral calculus.

$$V_{cone} + V_{sphere} = \frac{1}{2}V_{cylinder}$$

Subtracting V_{cone} from both sides, we get:

$$V_{sphere} = \frac{1}{2}V_{cylinder} - V_{cone}$$

Since $V_{cylinder} = \pi r^3$ and $V_{cone} = \frac{1}{3}\pi r^3$, by substitution, we get:

$$V_{sphere} = \frac{1}{2}\pi r^3 - \frac{1}{3}\pi r^3 = \frac{1}{6}\pi r^3$$

We now have the volume of a sphere of known diameter r. Since d is the normal representation for diameter, we can restate the volume formula in terms of both the diameter d and radius r recognizing that d = 2r:

$$V_{sphere} = \frac{1}{6}\pi d^3 \text{ or}$$

$$V_{sphere} = \frac{1}{6}\pi(2r)^3 = \frac{4}{3}\pi r^3$$

Archimedes next tackled the surface area of a sphere by noting this analogous relationship:

> … judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius.[10]

His first observation is comparing the area of a circle with the area of a triangle. Figure 5 reveals the connection he is making.

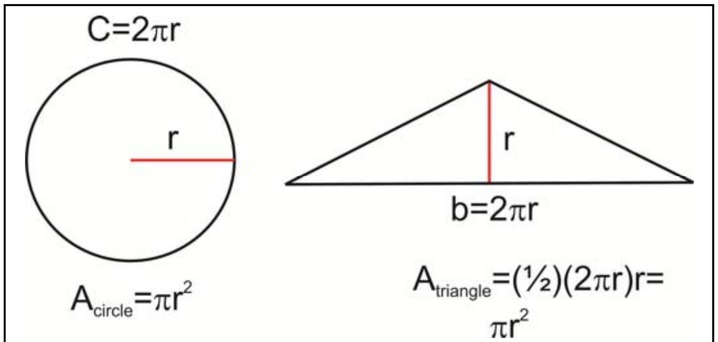

Figure 5

[10] Heath, pp. 20-21.

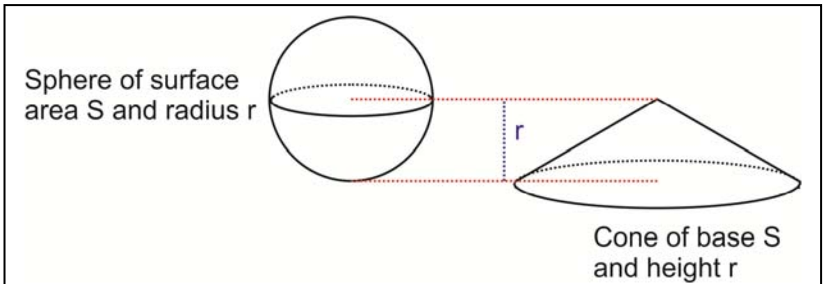

Figure 6

Figure 6 reveals the thinking of Archimedes applied to the sphere and cone. By his reasoning, we get:

$$\frac{4}{3}\pi r^3 = \frac{1}{3}Sr$$

Solving for S (multiplying both sides by 3 and dividing both sides by *r*), we get:

$$S = \frac{4}{3r}3\pi r^3 = 4\pi r^2$$

In a sphere, a great circle is defined as the locus of points that is the intersection of the sphere and a plane containing its center. The area of any great circle of radius *r* is therefore πr^2. The formula for the surface area of a sphere shows that this area is exactly four times the area of any great circle.

Anyone knowing the methods of the differential calculus will automatically see that the surface area of a sphere is the first derivative of the volume of the sphere; i.e., using functional notation:

$$V(r) = \frac{4}{3}\pi r^3$$

$$S(r) = V'(r) = \frac{dV}{dr} = 4\pi r^2$$

This means that the rate at which the volume of a sphere increases when the radius *r* increases *is the measure of the surface area of the sphere!*

Archimedes proved these formulas by a different method using more rigorous standards.[11] The beauty revealed by the genius of Archimedes is not only his rigor, but his ability to “see” connections and to reason to mathematical conclusions in terms of a mechanical starting point. By his method Archimedes is showing a connection between physical reality and mathematical thinking. In Archimedes, his “ignoble,” or “non-Platonic” work, reveals a *balanced* connection between concrete reality and abstract thinking (the metaphor is *intentional*). For the Biblical Christian, since God is the creator of both the physical world and the human mind, the Archimedean method turns of the fulcrum of the Creator God.

[11] For a summary analysis, see Meschkowski, pp. 16-22.