The Mathematics and Physics of Projectile Motion

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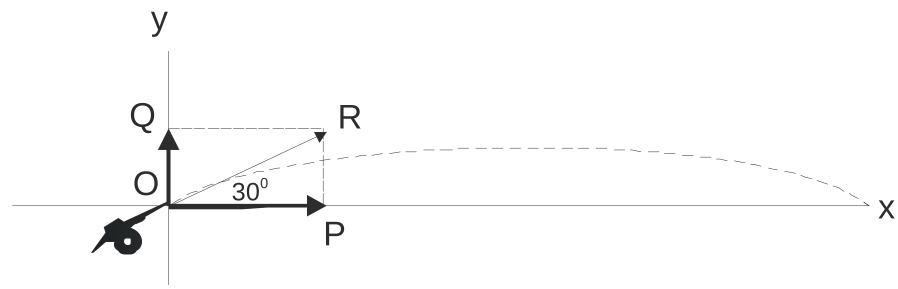
Problem



Suppose that a cannon is inclined at an angle of 30° to the ground. It fires a shell with a velocity of 1000 ft/sec. Let's try to analyze the subsequent motion of the shell by considering its horizontal and vertical motions separately (i.e., we shall obtain what are called *parametric formulas*). Note the parallelogram constructed (OQRP). The distance traveled in one second is represented by what physicists called the radius vector $\overrightarrow{OR}$.[1] The horizontal distance traveled in one second is represented by the radius vector $\overrightarrow{OP}$ and the vertical distance traveled in one second is represented by the radius vector $\overrightarrow{OQ}$. Let's determine these distances with the help of trigonometry.

Analysis

The ratio that represents cos 30° is $\frac{OP}{OR}$ or $\cos 30° = \frac{OP}{OR}$.

The ratio that represents sin 30° is $\frac{RP}{OR}$ or $\sin 30° = \frac{RP}{OR}$.

If the shell travels one second, then OR = 1000 ft. What about OP and RP? Since OR = 1000, then we can substitute as follows:

$$\cos 30° = \frac{OP}{OR} \Rightarrow 0.866 = \frac{OP}{1000} \Rightarrow OP = (1000)(0.866) = 866 \text{ ft/sec.}$$

$$\sin 30° = \frac{RP}{OR} \Rightarrow 0.5 = \frac{RP}{1000} \Rightarrow RP = (1000)(0.5) = 500 \text{ ft/sec.}$$

Since OQ = RP (opposite sides of a rectangle are equal), OQ represents the vertical velocity (*up* motion on the y-axis) in ft/sec. Hence, OQ = 500 ft/sec. We can now write an equation for *y* in terms of *t*; i.e., y = g(t) assuming that no force acts to accelerate or decelerate the horizontal motion. It is:

$$y = g(t) = 500t$$

Likewise, OP represents the horizontal velocity (*right* motion on the x-axis) in ft/sec. We can write another equation for *x* in terms of *t*; i.e., x = f(t). It is:

$$x = f(t) = 866t$$

These two equations, y = g(t) = 500t and x = f(t) = 886t are called parametric equations.[2] Since the force of gravity pulls the shell downward at a distance of $16t^2$ feet per second[3], we can extend the parametric equation y = g(t) = 500t as follows:

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[1] In geometry, $\overrightarrow{OR}$ is the symbol for the ray originating at point O and travelling through point R. Vectors, in physics, represent magnitude and direction. For example, a car traveling 60 miles/hr (*magnitude*) in the *direction* NE (northeast).

$y = g(t) = -16t^2 + 500t$

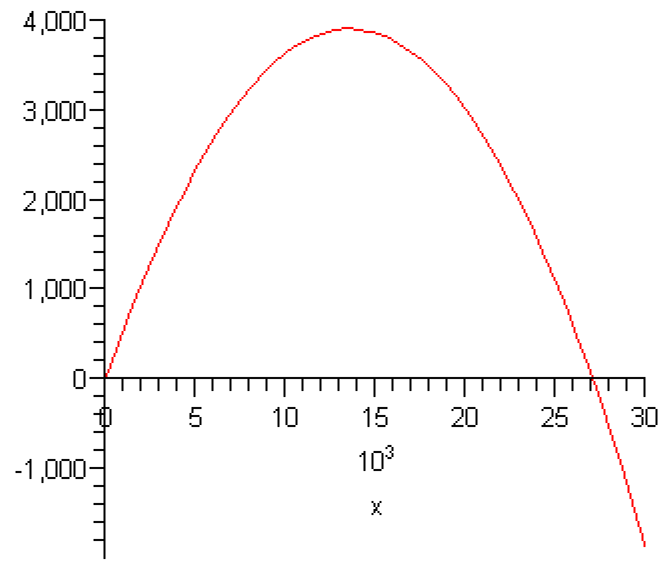
In vector physics, we represent the downward pull as a negative amount. The downward pull due to gravity is $16t^2$ (represented as $-16t^2$) and the upward thrust is $500t$ (represented by $+ 500t$).

Given these two equations, $x = f(t)$ and $y = g(t)$, we can generate an equation for $y = f(x)$.

Since $x = f(t) = 866t$, then $t = \frac{x}{866}$

By substitution, $y = f(x) = -16\left(\frac{x}{866}\right)^2 + 500\left(\frac{x}{866}\right)$ or $y = f(x) = -\frac{4x^2}{187,489} + \frac{250}{433}x$

This function, when graphed, is a parabola (one of the four conic sections[4]). From these equations, we can answer some very important "field" questions.



Field questions

Question 1: What is the *range* of the shell (i.e., how far from the starting point will the projectile strike the ground again)?

We know that when the shell hits the ground, the y-distance will be 0. Hence, we must solve this equation:

$0 = -16t^2 + 500t$

This explains why, when you took courses in algebra involving the solution of polynomial equations, you learned how to solve them (or, find the zeroes of the equation).

On the right side of the equation, the common factor is t:

$0 = t(-16t + 500)$

From this, we get two solutions. The first solution is $t = 0$. We derive the second solution as follows:

$-16t + 500 = 0$
$16t = 500$
$t = 31.25$

The two solutions are $t = 0, 31.25$. What does this mean? At $t = 0$, the projectile is in the cannon. At $t = 31.25$, the projectile hits the ground.

The distance traveled (on the x-axis) is governed by the equation $x = 866t$. At $t = 31.25$, $x = (866)(31.25) = 27,062.5$ ft (the range).

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[2] Technically, a parametric equation is one of two or more equations expressing the location of a point on a curve or surface by determining each coordinate separately.

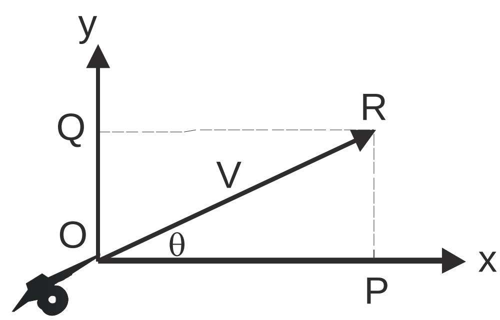
[3] Galileo Galilee (1564-1642) discovered this function by experimenting with balls rolling down inclined planes.

[4] The other three are the ellipse, the hyperbola, and the circle.

Question 2: What is the maximum height that the projectile reaches and how long will it take the projectile to reach it?

Before we answer these two questions, let's summarize our findings. We began with a specific problem. Let's generalize it. As before, we let t = time. The projectile leaves the cannon at t = 0 with a velocity V and an upward angle θ (Greek letter "theta"). We can divide V into its horizontal and vertical components: v_x and v_y. We let the distance travelled for one second (t = 1) to be OR. The vertical motion *up* is OQ = RP or Vsinθ. The horizontal motion *right* is OP = QR = Vcosθ.

Hence, the horizontal motion is defined by $v_x = V\cos\theta$ and the distance travelled in *t* seconds is defined by $x = (V\cos\theta)t = Vt\cos\theta$.



The vertical motion is defined by $v_y = V\sin\theta - 32t$ where 32t represents the acceleration downward (32 ft/sec²). In other words, gravity pulls the projectile downward at a velocity of 32t feet in *t* seconds. We let y = how far above the ground the projectile be at any time *t*. Hence $y = Vt\sin\theta - 16t^2$. This means that the projectile rises $Vt\sin\theta$ feet in *t* seconds and during this time, the action of gravity will pull the projectile downward $16t^2$ feet.

Since $x = Vt\cos\theta$, then $t = \dfrac{x}{V\cos\theta}$. Since $y = Vt\sin\theta - 16t^2$, then $y = \cancel{V}\sin\theta\left(\dfrac{x}{\cancel{V}\cos\theta}\right) - 16\left(\dfrac{x}{V\cos\theta}\right)^2 = \dfrac{\sin\theta}{\cos\theta}x - \dfrac{16x^2}{V^2\cos^2\theta}$. This equation, $y = \dfrac{\sin\theta}{\cos\theta}x - \dfrac{16x^2}{V^2\cos^2\theta}$, is in the form $y = ax - bx^2$ which, when graphed, results in a parabola.

We are now ready to answer our questions. When the projectile reaches its maximum height, its vertical velocity, v_y, will be zero. If $v_y = V\sin\theta - 32t$, this means that $0 = V\sin\theta - 32t$.[5] We solve for t:

$$32t = V\sin\theta$$
$$t = \frac{V\sin\theta}{32}$$

If, for example, the projectile is fired at an angle of 60° at an initial muzzle velocity of 800 ft/sec, maximum height will be reached in $\dfrac{800\sin 60°}{32} \approx 21.65$ seconds. Since $y = Vt\sin\theta - 16t^2$, then the maximum height of the projectile is $800(21.65)\sin 60° - 16(21.65)^2 \approx 7500$ feet.

Before we end this analysis of question 2, we can ask a couple of subsidiary questions. How long will it take the projectile to reach the ground again and what is the velocity of the projectile when it hits the ground?

We know that when the projectile strikes the ground, y = 0. So, we must solve the equation $0 = Vt\sin\theta - 16t^2$. We get:

$$0 = Vt\sin\theta - 16t^2$$
$$0 = t(V\sin\theta - 16t)$$

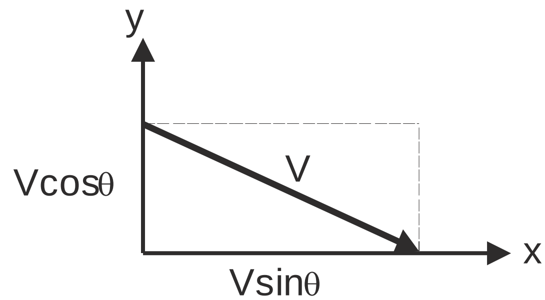
[5] Readers who know the calculus will recognize that $v_y = V\sin\theta - 32t$ is the first derivative of $y = Vt\sin\theta - 16t^2$ (the velocity function is the derivative of the position function). Setting the first derivative equal to 0 ($0 = V\sin\theta - 32t$) and solving for *t* will give us the time it takes to reach the maximum height.

The first solution is t = 0 (the initial release). For the second solution (projectile striking the ground at its terminal position), we must solve the equation $V\sin\theta - 16t = 0$. We get:

$$16t = V\sin\theta$$
$$t = \frac{V\sin\theta}{16}$$

If V = 800 and θ = 60°, then $t = \frac{800\sin 60°}{16} \approx 43.30$ seconds. Note: 43.30 = 2(21.65). It takes half of the total flight time to reach the maximum height.

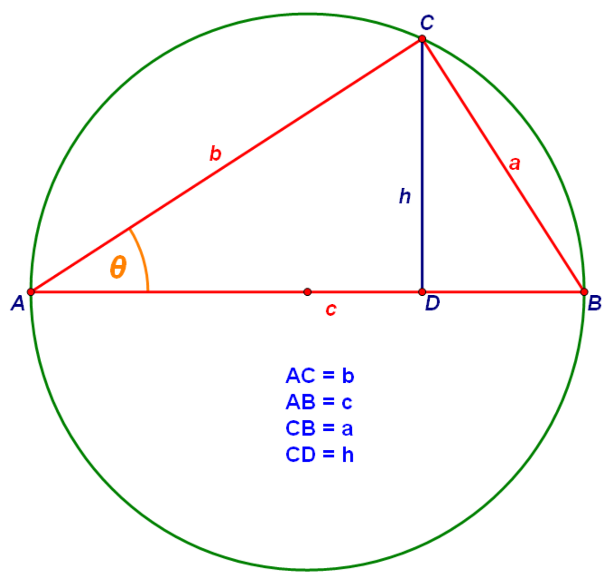
When the projectile strikes the ground, its vertical velocity is still $v_x = V\cos\theta$ and its horizontal velocity is $v_y = V\sin\theta$.[6] Since V is the union of these two components, then the projectile strikes the ground with precisely the same velocity as it started out!



Since $x = Vt\cos\theta$, we can easily determine the range. At t = 43.30 seconds, x = 800(43.30)cos60° = 17,320 feet.

Question 3: What angle inclination will result in the projectile travelling the maximum distance?

There should be a best angle of fire, meaning the angle that produces the longest range, and it is somewhere between 0° and 90°. We have already determined that when the projectile hits the ground, t = We have already determined that when the projectile hits the ground, $t = \frac{V\sin\theta}{16}$. Since $x = Vt\cos\theta$, then $x = \frac{V^2}{16}\sin\theta\cos\theta$. Since $x = Vt\cos\theta$, then $x = \frac{V^2}{16}\sin\theta\cos\theta$. We have to determine the value of θ that gives the maximum x (the maximum range).



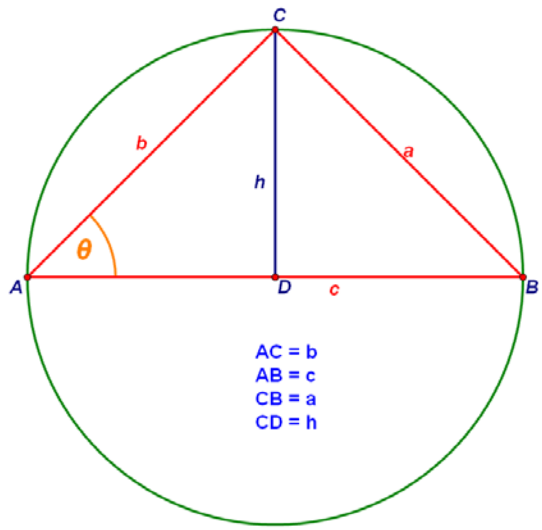
Since V is fixed, then x is maximum when $\sin\theta\cos\theta$ is maximum. Let's analyze this product. Inspect the figure at right. Since ΔABC is inscribed in a semicircle, we know, from Euclid's Geometry, that ΔABC is a right triangle. Next, we drop a perpendicular line segment from C to D. Hence, we have created another right triangle ΔADC. We set θ = ∠A. From trigonometry, we know:

$$\text{Right } \Delta ADC \Rightarrow \sin\theta = \frac{h}{b}$$

$$\text{Right } \Delta ABC \Rightarrow \cos\theta = \frac{b}{c}$$

[6] In the equation $v_y = V\sin\theta - 32t$, once the projectile strikes the ground, it is no longer moving. Hence, the term -32t cancels out and we get $v_y = V\sin\theta$.

Hence, sinθcosθ = $\left(\frac{h}{\cancel{b}}\right)\left(\frac{\cancel{b}}{c}\right) = \frac{h}{c}$. We note that *c* (diameter of the circle) is constant. Hence, the product, sinθcosθ, is maximum when *h* is maximum. When is *h* maximum? *h* is maximum when h = r (radius of the circle)! Hence, ΔADC is a *right isosceles triangle*. The angles opposite this triangle are equal (since the two legs are equal). We let θ = the measure of these two equal angles. Hence, 90° + 2θ =180° ⇔ 2θ = 90° ⇔ θ = 45°. Therefore, for any given initial velocity, we obtain the maximum range by firing at an angle of 45°.



Question 4: How does one reach point P where P < maximum range?

In real-life field applications, the projectile often needs to hit a target that is less than the maximum range. What should be the angle of fire? All we need to use is the equation x = $\frac{V^2}{16}\sin\theta\cos\theta$. Given the range x = OP, we solve this equation for θ.

How do we solve for θ? We make use of a double-angle identity from trigonometry: sin2θ = 2sinθcosθ.[7] Hence, sinθcosθ = $\frac{\sin 2\theta}{2}$. Our equation is now:

$$x = \frac{V^2}{16}\sin\theta\cos\theta = \frac{V^2}{16}\left(\frac{\sin 2\theta}{2}\right)$$

$$\frac{\sin 2\theta}{2} = \frac{16x}{V^2}$$

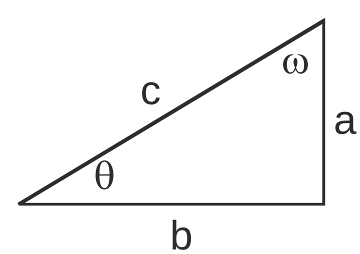
$$\sin 2\theta = \frac{32x}{V^2}$$

If x = 12,000 feet and V = 800 ft/sec, then $\sin 2\theta = \frac{32(12,000)}{800^2} = 0.6$. Hence:

$$2\theta = \sin^{-1}(0.6) \approx 36.87°$$

$$\theta \approx \frac{36.87°}{2} = 18.43°$$

This equation, x = $\frac{V^2}{16}\sin\theta\cos\theta$, tells us something else. In the right triangle, note that θ = 90° – ω (Greek letter omega) and ω = 90° – θ. This means that θ



[7] For the derivation of the formula, see any trigonometry textbook.

and ω are complementary angles (θ + ω = 90°). Note that $\sin\theta = \frac{a}{c} = \cos\omega = \cos(90° - \theta)$ and $\cos\theta = \frac{b}{c} = \sin\omega = \sin(90° - \theta)$. To summarize:

$\sin(90° - \theta) = \cos\theta$
$\cos(90° - \theta) = \sin\theta$

Given $x = \frac{V^2}{16}\sin\theta\cos\theta$, by substitution, we get:

$$x = \frac{V^2}{16}\sin\theta\cos\theta = \frac{V^2}{16}\cos(90-\theta)\sin(90-\theta) =$$
$$\frac{V^2}{16}\sin\theta\cos\theta = \frac{V^2}{16}\sin(90-\theta)\cos(90-\theta)$$

Hence, the angle of fire θ and 90° – θ (complementary angles) will yield the same range! In our example above, an angle of fire of 18.43° will yield the same range as an angle of fire of 71.57° (90° – 18.43°). One will be as far below the angle of maximum range (45° – 18.43° = 26.57°) as the other is above (71.57° – 45° = 26.57°)!

Such is one of the amazing revelations that mathematics gives us about the many nuances of the physics of motion, motion governed by the wisdom of God in Christ (Colossians 1:15-17; 2:1-3). These covenantal laws are treasures that man discovers by investigating the creation order (Proverbs 25:2). Classical physics, founded by men like Galileo, Kepler, and Newton, is a sequence of one fascinating revelation after another. This essay is only one example of these quantitative wonders.