Lessons from Ancient Greece: Number Theory

by James D. Nickel

The ancient or classical Greeks, particularly the School of Pythagoras (ca. 6^{th} century BC), explored many fascinating features revealed by investigating the structure of the counting numbers or {1, 2, 3, …}.[1] Using pebbles, they classified these numbers according to their geometric characteristics.[2] For example, they discerned a triangular pattern when they represented the counting numbers using pebbles (see Figure 1). Starting from 1, they noted this sequence, called the sequence of "triangular numbers":

1, 3, 6, 10, 15, 21, …
where
$1 + 2 = 3$
$1 + 2 + 3 = 6$
$1 + 2 + 3 + 4 = 10$, etc.

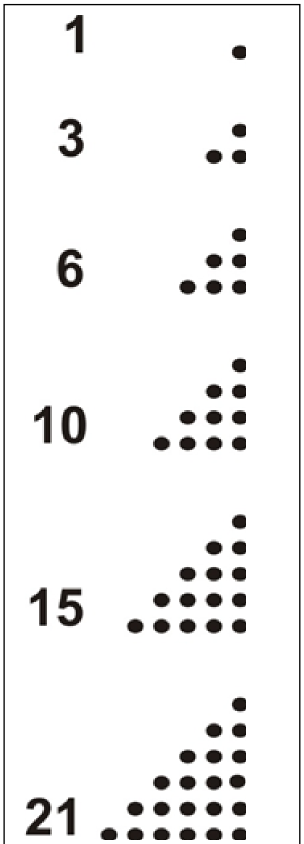

Figure 1

Let's investigate Figure 2.[3] The pebbles so arranged reveal that the sum of the first two triangular numbers (1 + 3) is a "square number"($1 + 3 = 4 = 2^2$), the sum of the second and third triangular numbers (3 + 6) is likewise a square number ($3 + 6 = 9 = 3^2$), etc. In general, the sum of any two consecutive triangular numbers is a square number.

The number of pebbles in these square arrays can be represented, in general, as n^2 using the algebraic variable n to represent each side.[4] Investigate Figure 3. Starting with one pebble (1^2), you must add 3 (1 + 1 + 1) pebbles to get the next square array of 4 (2 by 2). You must add 5 (2 + 1 + 2) pebbles to the square array of 4 (2 by 2) to get the next square array of 9 pebbles (3 by 3). Likewise, you must add 7 (3 + 1 + 3) pebbles to the square array of 9 (3 by 3) to get the next square array of 16 (4 by 4). In general, given a square array of n^2 pebbles, you must add $n + 1 + n = 2n + 1$ pebbles to get the next square array of $(n + 1)^2$ pebbles.

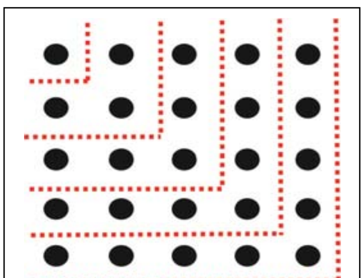
Figure 3

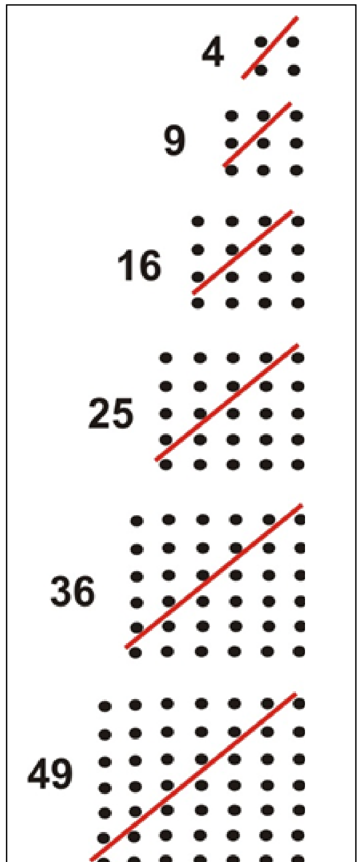

Figure 2

[1] Counting numbers are also known as natural numbers or positive integers.
[2] The *Latin* word for pebble (small stone) is *calculus* meaning "to count" or "to reckon." In medicine, the literal meaning is still apparent in the phrase "a calculous person," referring to one suffering from kidney stones. The Greek word for pebble is *psephos* (transliterated).
[3] The primary ideas of this essay have been taken from Herbert Meschkowski, *The Ways of Though of Great Mathematicians: An Approach to the History of Mathematics* (San Francisco: Holden-Day, 1964), pp. 2-5.
[4] The Greeks thought in these general concepts, but they did not write what was happening using modern symbolic algebra.

Summarizing, we get this algebraic identity:

$$n^2 + (2n + 1) = (n + 1)^2 \text{ or}$$
$$(n + 1)^2 = n^2 + (2n + 1)$$

From this same identity, we note that the differences between successive squares give us the sequence of odd numbers {1, 3, 5, 7, 9, …}.

n	$(n + 1)^2$	n^2	$(n + 1)^2 – n^2 = 2n + 1$
0	1	0	1
1	4	1	3
2	9	4	5
3	16	9	7
4	25	16	9
5	36	25	11
etc.	etc.	etc.	etc.

With this same equation, we can also derive an equation that will generate the numbers that satisfy the famous and ubiquitous[5] Pythagorean Theorem, a relationship about the lengths of the sides of a right triangle known to antiquity (i.e., before the age of the Classical Greeks). Rhetorically, in a right triangle, the square of the length of the hypotenuse[6] is the sum of the square of the lengths of the triangle's two legs. If c = the hypotenuse and a, b = the respective legs, we get:

Pythagorean Theorem

$$c^2 = a^2 + b^2$$

In the stamp pictured, we see these three squares and the equation is:

$$5^2 = 3^2 + 4^2 \text{ or}$$
$$25 = 9 + 16$$

The integers 3, 4, 5 are called *Pythagorean Triples* because they satisfy the Pythagorean Theorem.

Compare $c^2 = a^2 + b^2$ with $(n + 1)^2 = n^2 + (2n + 1)$. To make the connection between the Pythagorean Theorem and this algebraic identity, we must make 2n + 1 into a square. We do this as follows:

Let $m^2 = 2n + 1$. Solving for n, we get:

$$2n = m^2 – 1 \Leftrightarrow n = \frac{m^2 - 1}{2}$$

Hence, $n + 1 = \frac{m^2 - 1}{2} + 1 = \frac{m^2 - 1}{2} + \frac{2}{2} = \frac{m^2 + 1}{2}$

Substituting these new found values of n and n + 1 (in terms of m) into $(n + 1)^2 = n^2 + (2n + 1)$, we get:

[5] The Pythagorean Theorem is a unity that appears in a diverse number of branches of the mathematics tree: Geometry, Trigonometry, Probability, Calculus, and Einstein's Theory of Relativity to name only a few.
[6] The hypotenuse is the side opposite the right angle of the right triangle.

$$\left(\frac{m^2+1}{2}\right)^2 = \left(\frac{m^2-1}{2}\right)^2 + m^2 \text{ or}$$

$$m^2 + \left(\frac{m^2-1}{2}\right)^2 = \left(\frac{m^2+1}{2}\right)^2$$

Setting m = 3, 5, 7, 9, 11, etc., we can now generate integers (i.e., Pythagorean Triples), that satisfy $c^2 = a^2 + b^2$ (as demonstrated in the following table).

m	*a*	*b*	*c*
3	3	4	5
5	5	12	13
7	7	24	25
9	9	40	41
11	11	60	61

It may be possible that the Pythagoreans discovered the theorem named in their honor by such investigations into number theory and *not* by the comparison of areas (as seen in the stamp).

As we have noted, adding successive counting numbers (or positive integers) generates triangular numbers and adding successive odd positive integers generates square numbers or:

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2 \text{ for } n = 0, 1, 2, 3 \ldots$$

Adding successive even positive integers gives us this sequence:

$$2 + 4 + 6 + 8 + \ldots + 2n$$

Since these numbers are even, each shares the common factor 2. Factoring 2, we get:

$$2 + 4 + 6 + 8 + \ldots + 2n = 2(1 + 2 + 3 + 4 + \ldots + n)$$

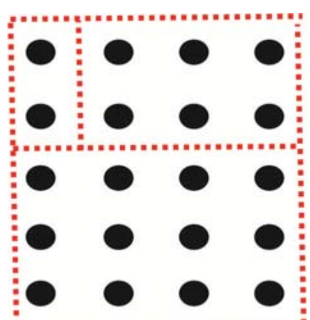

Figure 4

When n = 1, we get (2)(1) = 2 (see Figure 4). We can organize our pebbles 2 by 1 (or, as a rectangle). When n = 2, we get (2)(3) = 6 (or a 2 by 3 rectangle). When n = 3, we get (3)(4) = 12 (or a 3 by 4 rectangle). In general, summing *n* even numbers, we get n(n + 1) pebbles (or a "n" by "n + 1" rectangle). We get the following:

$$2 + 4 + 6 + 8 + \ldots + 2n = 2(1 + 2 + 3 + 4 + \ldots + n) = n(n + 1)$$

From this, we can derive a formula for finding the sum of the successive positive integers. Starting from:

$$2(1 + 2 + 3 + 4 + \ldots + n) = n(n + 1)$$

… and dividing both sides by 2, we get:

$$1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$$

The Pythagoreans concluded (by induction) that odd numbers generate *one and only one* form: a square (with all squares being similar). In contrast, the even numbers generate many, in fact an *unlimited* number, of forms of a rectangle (none are similar to each other). Since this ancient school believed that everything in the universe could be understood using ratios, here is confirmation! We have these ratios:

$$\frac{\text{odd}}{\text{limited}} \text{ and } \frac{\text{even}}{\text{unlimited}}$$

Building a system upon a monistic world view like this (i.e., all of reality can be understood in the context of counting numbers or the ratio of counting numbers) is suspect. It did not take long for the Pythagoreans to discover a number that did not fit into their presupposed world view.[7]

The fault of the Pythagoreans, i.e., building a system on unsupported generalizations, is still with us today. In mathematics, counterexamples can check conjectures that might be a little too bold. In the humanities or in theories of social order, however, it may take quite a long time for false ideas to be exposed. As Biblical Christians, the wisest world view position to embrace is one that never absolutizes the relative (i.e., anything in the created order) and never relativizes the absolute (i.e., the Biblical God). *The fear of the Lord is the beginning of wisdom and knowledge* (Proverbs 1:7, 9:10). Nothing in the created universe (not human reason, not logic, not mathematics, not nature) can provide the basis for knowing truth. The only true foundation for knowledge, the only true starting point that both guides and provides an eternal safety net for all aspects of human endeavor, is *trust* in the infinite, person, and Triune God revealed in Holy Scripture.

[7] The discovery of $\sqrt{2}$ as a number that can be viewed geometrically (e.g., the length of the diagonal of a unit square) but not a number that can be written as the ratio of two positive integers revealed significant cracks in the Pythagorean worldview foundation.