Of Stones, Summation, and Academic Sterility

> Society does not speak of the matter of the one and the many; most people are ignorant of the problem, even though it is basic to all life and thought. But because of man's failure to solve the problem, society is caught in the continuing tensions of alternating anarchy and totalitarianism, between anarchic individualism and anarchic collectivism. Philosophy has in recent years abandoned the battlefield for the academic sterilities of logical analysis. If there is to be any kind of Christian reconstruction, then, in every area of thought, the philosophy of Cornelius Van Til is of critical and central importance.[1]
>
> Rousas John Rushdoony

The Latin word for stone is *calculus* (the diminutive of the Latin *calx*, meaning "stone").[2] Small stones (pebbles) were used in ancient times on a counting board or abacus. Because of this, the word carries within it the meaning of "counting, summing, or calculating." Calculus once was a common designation (until the early 19th century) that denoted all the branches of mathematics (e.g., arithmetical calculus, algebraical calculus, etc.). In modern times, *the calculus* is a branch of mathematics that deals with the measure of motion or any change at an instant (termed the *differential calculus*) and the inverse of this operation, namely the calculation of areas bounded by curves (termed the *integral calculus*).

Mathematician Richard Courant (1888-1972), in the Foreward to Carl Boyer's (1906-1976) important study entitled *The History of the Calculus and its Conceptual Development*, identifies the academic sterility that too often accompanies the teaching of this subject:

> Differential and Integral Calculus and Mathematical Analysis in general is one of the great achievements of the human mind. Its place between the natural and humanistic sciences should make it a singularly fruitful medium of higher education. Unfortunately, the mechanical way in which calculus sometimes is taught fails to present the subject as the outcome of a dramatic intellectual struggle which has lasted for twenty-five hundred years or more, which is deeply rooted in many phases of human endeavors ...[3]

I want to accomplish two goals in this essay.[4] First, I hope to explicate the key point of transition in the historical development of this great human achievement. I will show that at this key point, men embracing the Christian World View acted as a catalyst in the intellectual transformation of the subject. Second, as an application of the thought of Rousas John Rushdoony (1916-2001), I also hope to show that the matter of the one and the many is basic to an understanding of the central conception underlying the interplay between the differential and integral calculus, namely the Fundamental Theorem of the calculus. When the academe rejects the *wonder* of these two truths in favor of a pure "mechanical" teaching of the subject, the logical conclusion, as inferred by Rushdoony in the prefatory quote, is the dry *sterility* of logical analysis.

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1 Rousas John Rushdoony, *The One and the Many: Studies in the Philosophy of Order and Ultimacy* (Fairfax, Virginia: Thoburn Press, 1978), 362-363.

2 In medicine, the literal meaning is still apparent in the phrase "a calculous person," referring to one suffering from kidney stones.

3 Carl Boyer, *The History of the Calculus and its Conceptual Development* (New York: Dover Publications, [1949] 1959), Foreward.

4 Much of the text of the essay is an extracted summary from the section entitled "The Mathematics of Change at an Instant" in James Nickel, *Mathematics: Building on Foundations*, unpublished manuscript.

Conceptual Fortifications of the Calculus

The overall structure of the calculus is founded upon several simple, yet profound, ideas. In the introduction to his beguiling and instructive book about the calculus, David Berlinksi states, "The body of mathematics to which the calculus gives rise embodies a certain swashbuckling style of thinking, at once bold and dramatic...."[5] This way of mathematical thinking finds its historical roots in the attempt to get a handle on the nature and nuances of motion. In specific, the deliberation of accelerated motion in space (e.g., consider the ever increasing speed, i.e., variability, of a stone as it falls from its release point) and motion at an instant of time (e.g., consider the speed of that falling stone at precisely two seconds after its initial release). The analysis of spatial motion in time, the where and the when, requires three ancillary ideas: (1) the mathematical function[6] which requires the use of (2) symbolic algebra and an acceptance of (3) the arithmetical real number continuum. Technically, this real number continuum is the union of the set of rational numbers with the set of irrational numbers.[7] Two mathematical tools, the Cartesian coordinate plane (x-y axis) and a real number, or real value, function, subordinates the created space/time continuum (Genesis 1:1) to man's investigatory powers (an application and extension of the dominion mandate of Genesis 1:26-28).[8]

Undergirding all these mathematical conceptions is the concept of the nature of infinity (as it relates to the real number continuum) and the infinitesimal (the calculation of ever decreasing values that approach zero as a limit). Note these remarks about this continuum made by the mathematician Tobias Dantzig (1884-1956):

> Indeed, whether we use a ruler or a weighing balance, a pressure gauge or a thermometer, a compass or a voltmeter, we are always measuring what appears to us to be a *continuum*, and we are measuring it by means of a graduated *number scale*. We are then assuming that there exists a perfect correspondence between the possible states within this continuum and the aggregate of numbers at our disposal; … Therefore, any measuring device, however simple and natural it may appear to us, implies the whole apparatus of the arithmetic of real numbers: behind any scientific instrument there is the master-instrument, arithmetic, without which the special device can neither be used nor even conceived.[9]

The derivative is a mathematical technique used to quantify (or calculate) the speed of the falling stone *at an instant of time*.[10] This process produces a new or derived function. The inverse of this process, called anti-

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[5] David Berlinksi, *A Tour of the Calculus* (New York: Vintage Books, 1995), xiii.

[6] For example, $s = 16t^2$ and $v = 32t$. The first function relates distance, denoted as *s* (the abbreviation of the Latin word for distance; i.e., *spatium*) to time (denoted as *t*). In the physics of Newtonian mechanics, this "position function" describes the position (or distance) of an object dropped from a given height (free-fall motion) at a specified time. The second function, the "velocity function," connects velocity (a vector quantity representing speed and direction), denoted as *v*, with time. The coefficients 16 and 32 reflect British imperial measurement. 32 represents a measurable (via experiments with pendulae and watches) acceleration constant; i.e., $a = 32$. This means that the Earth exerts a force on a falling object giving it an acceleration of 32 feet/second for every second of its free fall (i.e., 32 feet/sec^2). Amazingly, this number can also be derived mathematically by observing the Moon and analyzing its motion and by assuming gravity to be the common cause, both of the motion of the Moon around the Earth, and the fall of an apple to the ground. See J. M. Knudsen and P. G. Hjorth, *Elements of Newtonian Mechanics* (New York: Springer-Verlag, 1995), 1-8.

[7] The rational numbers (e.g., fractions), when plotted on a number line, give the line a sense of being "everywhere dense." That is, between any two rational numbers, you can calculate another rational number (by finding the average between the original two). In fact, you can calculate an infinite number of rational numbers between any two rational numbers. In spite of this compactness, *gaps appear throughout the infinite extent of the number line*. These holes cannot be represented by a rational number. The irrational numbers (e.g., $\sqrt{2}$ and π) "fill up" these gaps making the real number line a true continuum.

[8] This subordination effectively maps the space/time continuum into a two-dimensional Cartesian plane (the x-axis represents time and the y-axis represents space).

[9] Tobias Dantzig, *Number: The Language of Science* (New York: Doubleday Anchor, [1930] 1954), 245-246.

[10] For example, the position function describing the motion of a falling object is $s = 16t^2$ where s = position and t = time. The derivative of this function (denoted as ds/dt or s′) is 32t which turns out to be the velocity function; i.e. v = ds/dt (means "small change is *s* over small change in *t*") = s′ (read "s prime") = 32t. At 2 seconds, the object is falling downward at a velocity of 64 feet per second. In other words, its instantaneous rate of change at t = 2 is 64 feet per second. The derivative of the velocity function

differentiation or taking the integral, uses the derived function to measure the distance the stone travels over a quantifiable period of time (e.g., 4 seconds).[11] This distance is also the same as the area under the geometric curve that represents this derived function.[12]

In summary, the particularity of the study of the physical nature of motion led to the rise of these mathematical techniques. Extending these procedures universally (generalizing differentiation and integration), scientists and mathematicians discovered to their amazement that the same consanguinity that unites speed with area also holds between *any* idea *similar* to speed and area; i.e., any physical situation that can be denoted by a real value function. Thus, a universe of applications came under the potent sway of the calculus.

Contributions and Shortcomings of the Ancient Greeks

The concept of motion was a headache to the ancient Greeks. It was their failed attempt to get their hands on motion that portended the invention of the calculus many centuries later. An example of this motion conundrum is the perplexing paradoxes of Zeno (495-430 BC). For example, in the Dichotomy paradox, Zeno proved that motion was impossible by the following reasoning. Imagine that you are going to run a race. As you approach the starting line, you think, "Before I get to the finish line, I must pass the halfway mark. Before I reach the halfway mark, I must pass the quarter mark. Before I reach the quarter mark, I must pass the one-eighth mark, and so on, indefinitely. Hence, I can *never* start running!" Because of Zeno's propositions, Greek philosopher Aristotle (384-322 BC) concluded, "Nothing can be in motion in a present ... Nor can anything be at rest in a present."[13] In this inability to corral motion conceptually, the Greeks rejected a key component of the calculus, i.e., the consideration of the notion of instantaneous rate of change.

Although the ancient Greeks anticipated the calculus, they could not advance the subject to its fullest for two reasons. First, they had trouble with one concept, namely *infinity*. The Greek engineer Archimedes (ca. 287-212 BC) started the climb to the infinitesimal heights of the calculus when he calculated the lower and upper limits of π (the ratio of the circumference to the diameter of a circle) by a means called the *method of exhaustion*.[14] Archimedes *exhausted* the circumference of a circle with inscribed and circumscribed regular polygons and he calculated his solution in terms of *finite* sums. The word *infinity* never appeared in any of his arguments. In the case of Zeno, he concluded that motion was impossible because he could not accept the fact that an *infinite* sum of numbers could converge to a limit. The *convergence of an infinite series to a limiting value*, a foundational concept of the calculus, resolves Zeno's conundrums.[15] For Aristotle, anything beyond the

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(denoted as dv/dt or v′), or the second derivative of the position function (denoted as d^2s/dt^2 or s″) gives us the acceleration constant; i.e., $a = d^2s/dt^2 = s'' = dv/dt = v' = 32$.

[11] Given the velocity function, v = 32t, we "integrate" it or take its "anti-derivative" to get the position function, $s = 16t^2$. The distance the object falls between t = 1 and t = 5 (4 seconds) is calculated as follows: $16(5)^2 – 16(1)^2 = 16(25) – 16 = 400 – 16 = 384$ feet.

[12] In the example cited in footnote 11, 384 is the area under the curve v = 32t from t = 1 to t = 5.

[13] Aristotle, *Physica* VI. 234a. Cited in Boyer, 43.

[14] With this method, Archimedes followed the lead of Eudoxus (408-355 BC), a Greek astronomer and founder of Euclidean geometry.

[15] For example, consider 11 1/9.

$$11\frac{1}{9}=\frac{99}{9}+\frac{1}{9}=\frac{100}{9}=11.1111\ldots$$

11 1/9 is *exactly* equal to 11.1111. Mathematicians express this by saying that the sequence of partial sums,

$$11, 11.1=11+\frac{1}{10}, 11.11=11+\frac{1}{10}+\frac{1}{100},\ldots$$

tends to the limit 11 1/9, or by saying the infinite series,

$$11+\frac{1}{10}+\frac{1}{100}+\frac{1}{1{,}000}+\ldots$$

is *convergent* and its sum, or limiting value, is 11 1/9.

power of comprehension (i.e., common sense or intuitively reasonable) is beyond the realm of reality.[16] It is the *transcendent* nature of infinity that rattled the Greek mind. By transcendent, I mean that the concept of infinity goes beyond the limits of human reason (and common sense, for that matter) and for that reason the Greeks were horrified by it (in Latin, *horror infiniti*). The Greek worldview erred in one of two ways: either they (1) absolutized number (in the case of Pythagoras[17]) or they (2) absolutized reason (in the case of their philosophers). When a culture absolutizes (or deifies) any aspect of God's creation, *then nothing can transcend that deity*. Since the concept of infinity transcended Greek deity (i.e., human reason or common sense), then the Greeks, shrinking before its silence, swept this intruder under the proverbial rug.[18]

Second, since the Greeks connected form to number in the context of plane and solid geometry, their understanding of algebra was inadequate. Because they were limited to rhetorical algebra (Greek mathematics also included no general concept of number), they did not develop a collection of symbols and a set of generic rules with which to operate upon these symbols. The reason why they failed to embrace a truly symbolic algebra is again worldview related. Their *static* view of the world is reflected by their geometric commitments. In plane geometry, all lengths have fixed (or static) magnitudes. In symbolic algebra (fully developed in Christianized Western Europe), letting x be equal to a variable quantity presupposes a *dynamic* view of the world. That is, the *domain* that x can assume can *range* across the continuum (in the case of motion, this continuum is time). Greek geometry, with its static line segments and angles (which serves its intended purpose quite well), is alien to the dynamics of the continuum. Greek geometry cannot express relations among *variable* quantities. The Greeks could not connect "things in motion" and variability to quantitative explanations (i.e., mathematical functions).

In summary, Greek mathematics, although anticipating the calculus, only adumbrated the subject because of three key constraints. First, the Greek idea of motion (tied to Greek theology, as we shall see shortly) rejected the possibility of instantaneous rate of change (basic to the derivative). Second, the Greeks did not grasp the nature of the real number continuum. They never carried anything to "infinity" or "to the limit" as the methods of the calculus do on a regular basis. Third, the Greeks could not connect "things in motion" with a mathematical symbolism (algebra) and a variability concept to a mathematical function.

Medieval Heritage

Close to two millennia after Archimedes, François Viète (1540-1603), the founder of symbolic algebra, in a work on trigonometry (published in 1593), discovered a remarkable formula involving π:

$$\frac{2}{\pi}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdot\ldots$$

Note the ellipsis (i.e., …) indicating that $2/\pi$ is an *infinite* product. As far as mathematics historians can ascertain, this was the *first* time an infinite process was explicitly written as a mathematical formula.[19] Viète's use of the ellipsis signaled an acceptance by the mathematical world of the infinite process and opened this method to widespread use. What conceptual changes engendered this use? Why?

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16 Boyer, 40. Aristotle embraced a metaphysic called "common sense reality."

17 Contra Pythagoras (ca. 582 - ca. 500 BC) who believed that number *generated* the particularity of things, number generates nothing; *number only reports on the particularity of things*.

18 For example, consider the case of the discovery of irrational numbers (geometrically represented as the diagonal of a unit square). The measure of this diagonal, an irrational number, is denoted today as $\sqrt{2}$ (its decimal expansion does not end and it is without pattern). The Greeks also denoted this number as incommensurable (unmeasureable). Tradition states that the Pythagoreans were at sea and they threw one of their members, a man named Hippaus, overboard for producing an element in the universe that could not be generated by number (either a whole number or whole number ratios; i.e., fractions).

19 Eli Maor, *Trigonometric Delights* (Princeton: Princeton University Press, 1998), 140.

Viète, from France, can cite the medieval schoolman Nicole Oresme (ca. 1323-1382), not only as a fellow countryman, but also as one of his mathematical forefathers. After commenting upon the syncopated algebra of Diophantus,[20] Carl Boyer denotes the important conceptual changes that occurred in the Middle Ages:

> ... in order that his [Diophantus – J.N.] work might be associated with geometrical results and later serve as the suitable basis for the calculus, it had to be made more completely symbolic, the concept of number had to be generalized, and the ideas of variable and function had to be introduced. During the Middles Ages the interest of the Hindus and the Arabs in algebraic development, and the attack by the Scholastic philosophers upon the problems offered by the continuum, were to supply in some measure the background for these changes.[21]

Oresme was a bishop of Lisieux and professor at the Sorbonne, an adjunct of the University of Paris, one of the premier medieval universities. It was at this university that medieval scholastics[22] embraced a profound and new concept concerning the theory of motion. It was during the 13th and 14th centuries that the Greek scientific corpus, primarily the works of Aristotle, found their way into the purview of medieval scholastics (via the agency of Arabic scholars). Aristotle believed that the universe was uncreated and eternal (time being understood as cyclical in nature[23]). Sorbonne professor Jean Buridan (1300-1358), Oresme's mentor, and his fellow scholars, including Thomas Aquinas (1225-1274), rejected this thesis as unBiblical and not in accord with Christian tenets, i.e., that God the Father Almighty created the universe "in the beginning" (time being understood as linear in nature[24]). The universe of the Christian is an interconnected entity of particulars both created and finite. In this Biblical context, Buridan considered what Aristotle postulated about motion.

Aristotle's theory of motion was lashed to his *a priori* (and deterministic) pantheistic emanationism. According to Aristotle, the universe consisted, as you recall the Greek geocentric arrangement, of a series of concentric, crystalline spheres with the Earth at the center. The universe moved because its highest sphere, the sphere of the fixed stars, was in a sort of "contact" with a "divine motor" that Aristotle called the "Unmoved or Prime Mover." Aristotle believed that it was through eternal contact with this "undefined" Prime Mover that the universe continued in its eternal motion (or rotation). Aristotle also pronounced (again *a priori*) that the celestial spheres reflected perfect motion. To him (and to the rest of his Greek collaborators), the circle is perfection and the heavenly rotations reflect perfect circular motion. Motion on the terrestrial realm, i.e., the Earth, being at the farthest point from the perfect motive power (or emanation) of the Prime Mover was, by its distance from the source of that emanation, *imperfect* (or, in the state of *partial disorder*). This cosmological belief put the empirical study of terrestrial motion in limbo and necessitated its attendant Zenonian incongruities.

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[20] The Alexandrian mathematician Diophantus (ca. 250) developed a syncopated algebra that used abbreviations for certain recurring quantities and operations.

[21] Boyer, 60. On a positive note, the Hindus generalized the number system, released arithmetic from its geometrical representation, embraced zero as a number, and developed the base ten number system (along with its attendant positional notation). The Arabic culture was highly eclectic. Their algebra, although one step from the symbolic algebra of Viète, still lacked a generalized conception. Like the Hindu culture, the Arabs did not care to speculate on matters incommensurable, continuous, indivisible, or infinite. The Arabs served an indispensable role of transmitting Greek works and the Hindu number system to Medieval Europe.

[22] Scholastic theologians have received bad historical press. Although they did err in some aspects of theology, they deposited an indispensable heritage for Western civilization (especially in science and mathematics).

[23] Time is in an "infinite loop" where the code of history continuously repeats itself. No long-term commitment to progress can be established with this treadmill (we are running but going nowhere) view of history.

[24] Time is "sequential" where the code of history is ordained in terms of God's eternal decrees. The sequential nature of time is inherent in the Christian creeds. They start from creation, progress to the Incarnation, declare the Christ as the Lord and Judge of history (and His holy congregation operative in that history), and end with the Second Advent, the general resurrection, and the eternal state. Long-term commitment to growth and development can be established with this structured (we are walking toward a goal) view of history.

Buridan presupposed, based upon Biblical revelation, an *absolute beginning* for physical motion. He understood the universe as being *distinct* from its Creator. To assume the contact that Aristotle postulated was pantheism (i.e., the universe, in its contact with the Prime Mover, was intimately connected to and part of that Source of movement). Buridan queried, "If the universe is distinct from its Creator, then how do we account for the movements of the celestial orbs?" At this point, Buridan's genius came into play, a genius motivated consciously by his belief in the tenets of the Christian God. He stated that at the moment of creation, God imparted motion to the universe and in that motion He established general influences (ordinances) that governed its continued motion. He said:

> When God created the world, He moved each of the celestial orbs as He pleased, and in moving them He impressed in them impetuses which moved them without His having to move them any more except by the method of general influence whereby He concurs as a co-agent in all things which take place; ... these impetuses which He impressed in the celestial bodies were not decreased nor corrupted afterward, because there was no resistance which would be corruptive or repressive of that impetus.[25]

This innovative thought generated a fruitful heritage. First, it led to an eventual full-fledged study of dynamics in a quantifiable context. The medieval theorizing on motion reflected a change in worldview: from the static view of the Greeks to a dynamic view (that led to the development of symbolic algebra). Second, it laid the conceptual groundwork for Isaac Newton's (1642-1727) first law of motion (also called the law of inertial motion or the law of impetus).[26] Newton, considered a co-inventor of the calculus, developed this mathematics in the context of the quantified study of motion (i.e., position, velocity, and acceleration). Third, it led to the conviction that the physics of the terrestrial and celestial realm could be united by a common law. This persuasion was contra to Aristotle's radical separation of these two domains into the perfect (regular) celestial realm and the imperfect (irregular) terrestrial realm. The Law of Universal Gravitation, developed by Newton, reflects that union. Fourth, it made acceptable the notion of instantaneous velocity (a concept foreign to Greek thinking).[27] Finally, many, if not most, of the medieval scholars in France, Italy, and England entered into lively discussions about infinity and the continuum.[28] Why? Since infinity (meaning "without bounds" theologically) was an attribute of the Biblical God, medieval theologians, unlike Greek philosophers, were not afraid of discussing its potential and actual nature and its implications.[29]

Concerning the meditations of the scholastic philosophers, mathematics historian Howard Eves observes:

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[25] Cited in Marshall Clagett, *The Science of Mechanics in the Middle Ages* (Madison: University of Wisconsin Press, 1959), 536. Buridan expanded this idea stating that a body, once set in motion, continues to move because of an internal tendency (momentum) which it then possesses. This idea is contra Aristotle, who stated that an external force (e.g., air), in direct contact with receding and irregular emanations from the Prime Mover, pushes a body along.

[26] For further details regarding the law of inertia, see James Nickel, "The Incarnation and Modern Science," *The Chalcedon Report* (December 2002, No. 447).

[27] Boyer, 73.

[28] For example, see Thomas Bradwardine (1290-1349), theologian at the Merton School (Oxford University), "On the Continuum." In Clagett, 231-233. Bradwardine used the Latin word *integrari* (making up the whole) stating that a continuous magnitude is "made up" of an infinite number of indivisibles of the same kind. Gottfried Wilhelm Leibniz (1646-1716), at the behest of his friend Johann Bernoulli (1667-1748), adopted this word to denote integration (the summing up of an infinite number of indivisibles to make a whole).

[29] In these discussions, a direct tie to mathematics was not made. Yet, by not shying away from the infinite, the medieval schoolmen parted with Aristotle's "common sense" realism and his unequivocal statement that mathematics and motion do not mix. See Aristotle, *The Metaphysics* (1061), trans. John H. McMahon (Amherst: Prometheus Books, 1991), 226. These men were willing to embrace speculations about the nature of infinity, the infinitesimal, and continuity that eventually found its way into the mathematical corpus. For example, Richard Suiseth (ca. 1337), known as "The Calculator," recognized that the infinite and the finite require different arguments. In 1632, Galileo Galilei (1564-1642) duplicated Suiseth's ideas almost verbatim saying, "... we cannot speak of infinite quantities as being the one greater or less than or equal to another." In *Dialogue Concerning Two New Sciences*, trans. Henry Crew and Alfonso de Salvio (New York: Dover Publications, [1914] 1954), 31.

> … [they] led to subtle theorizing on motion, infinity, and the continuum, all of which are fundamental concepts in modern mathematics. The centuries of scholastic disputes and quibblings may, to some extent, account for the remarkable transformation from ancient to modern mathematical thinking.[30]

The application of the concept of infinity to the study of change in motion is the sum and substance of the calculus. Carl Boyer comments about the impact of the input of the scholastics in this area:

> The blending of theological, philosophical, mathematical, and scientific considerations which has so far been evident in Scholastic thought is seen to even better advantage in a study of what was perhaps the most significant contribution of the fourteenth century to the development of mathematical physics … a theoretical advance was made which was destined to be remarkably fruitful in both science and mathematics, and to lead in the end to the concept of the derivative.[31]

Oresme, following Buridan's lead, initiated several key concepts that foreshadowed many of the methods of the calculus. In algebra, he invented one key component, namely fractional powers.[32] He also pioneered mathematical methods that dealt quantitatively with change and rate of change.[33] In other words, Oresme considered answers to questions like the following: If an object moves with varying speed, how far will it move in a given time? If the temperature of a body varies from one part to another, how much heat is there in the entire body? Today, we know that the calculus answers these queries. Pioneers like Oresme did not know this. He had to explore new avenues in order to develop some form of analysis. In order to visualize a object moving at variable velocity, Oresme associated "instants" of time within an interval with points on a horizontal line segment (denoted as a "line of longitudes"). To each of those points he associated (in a plane) a vertical line segment (denoted as "latitude"), the length of which represented the speed of the object at that instant of time (what we now know as the derivative of the position function[34]). Drawing a line connecting the extremities of these latitudes, he was able to represent the variation in velocity with time. This is one of the earliest instances in the history of mathematics of what we now call the "graph of a function."[35] He then determined, and he was the first to notice this, that the area under this graph represented a physical quantity; i.e., the distance covered because this area represents the summation of all the small increments in distance corresponding to the instantaneous velocities. In effect, Oresme was integrating a function over a range of time values. Galileo Galilei (1564-1642) used the same sort of reasoning 250 years later.[36]

"A Miracle of Mankind"

In his autobiography, historian Arnold Toynbee (1889-1975) notes the significance of the calculus:

> Looking back, I feel sure that I ought not to have been offered the choice [whether to study Greek or calculus – J.N.] … calculus ought to have been compulsory for me. One ought, after all, to be initiated into the life

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[30] Howard Eves, *An Introduction to the History of Mathematics* (New York: Holt, Rhinehart and Winston, [1953, 1964, 1969] 1976), 213.
[31] Boyer, 70-71.
[32] For example, in symbolic algebra, $x^{3/2}$. Oresme, of course, did not express the exponents in this format; he merely introduced the concept.
[33] Nicole Oresme, "On the Configuration of Qualities." In Clagett, 347-361.
[34] Given the position function $s = 16t^2$, then the derivative is the velocity function; i.e., $v = s' = 32t$. In terms of coordinate geometry, the derivative of the position function at a certain point on the graph is the slope (the ratio of rise to run) of the line tangent to that point. In the context of motion, the slope of this tangent line also represents the instantaneous velocity at that point. In other words, an object in motion at any particular instant is moving as fast as it would be moving if it were moving along a straight line.
[35] The Frenchman René Descartes (1596-1650), schooled in a tradition that can be traced directly to Oresme and considered the founder of this type of analysis (called analytical geometry or coordinate geometry or Cartesian geometry), never honored Oresme as a pioneer in the graphical representation of a function.
[36] In more than one instance (including this one), Galileo owes his apparent "originality" to his medieval forebears. Like Descartes, he never cited the source of his ideas. And, I might add, science historians (with some notable exceptions) also decline to divulge to their readers this medieval heritage.

> of the world in which one is going to live. I was going to live in the Western World … and the calculus, like the full-rigged sailing ship, is … one of the characteristic expressions of the modern Western genius.[37]

Using Toynbee's words, the "characteristic expression of the modern Western genius" finds its roots in the Biblical theology of the medieval schoolmen. Without the Biblical view of the infinite, Creator God, Western man (with his culture impacted by the Gospel of Christ) could never have embraced the infinitesimal nature of the calculus. Christ is the Savior, in an historical sense, and Lord, in an epistemic sense, of science and mathematics.[38]

For the calculus to be successful launched, it needed an acceptance and use of the concept of infinity, symbolic algebra, coordinate geometry (along with trigonometry), and the presence of reliable measuring tools (including mechanical clocks).[39] All of these ideas and instruments were on the "launching pad" in Western Europe at the dawn of the 17th century. The scientists of this century were trying to solve the following problems (some of which were initiated centuries before by the scholastics) and the calculus was perfected (in a pragmatic sense) to quantify and resolve the following:

(1) The motion of celestial bodies.
(2) The study of projectile motion and rays of light striking the surface of a lens (telescopes). To do this, a method was needed for the determination of tangent lines to various curves.
(3) Optimization or maxima and minima problems. In warfare applications, a method was needed to determine the maximum range and angle of elevation for artillery cannon. Concerning planetary motion, a method was needed for determining the maximum and minimum distances of a planet from the Sun.
(4) The study of lengths of curves and the areas and volumes of figures bounded by curves and surfaces (e.g., the Earth).

> The *integral* or the area *A* under the curve denoted by the function f(x) (read "f of x") between the lower bound *a* and the upper bound *b* is formally defined as:
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> $$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(m_i)\Delta x = \int_a^b f(x)dx$$
>
> *lim* stands for "limit." $\sum$ means "summation or aggregate." Δx (read "delta x") means "infinitesimally small difference in x." $f(m_i)\Delta x$ represents the area (length times width) of an infinitesimally small rectangle. We are finding the aggregate of an infinite number of such rectangles. At the limit, $f(m_i)$ becomes a point on the curve f(x) and each Δx interval can be represented now by dx (which stands for "a small change in x"). This is why $f(m_i)\Delta x$ becomes f(x)dx.
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> The *derivative* of the function f(x) is written f′(x) and is formally defined as:
>
> $$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$
>
> This procedure effectively calculates a specific number; i.e., the slope (the ratio of rise to run) of the line tangent to the curve represented by f(x) at any point x.

Many men of this century, basing their work consciously or unconsciously upon medieval heritage[40], contributed to the furthering evolvement of the calculus. The original contributors were Pierre de Fermat (1601-1665), René Descartes (1596-1650), Blaise Pascal (1623-1662), Gilles Persone de Roberval (1602-1675), Bonaventura Cavalieri (1598-1647), Isaac Barrow (1630-1677), James Gregory (1638-1675), Christian Huy-

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[37] Arnold Toynbee, *Experiences* (New York: Oxford University Press, 1969), 12-13.
[38] See Stanley L. Jaki, *The Savior of Science* (Grand Rapids: Eerdmans, 2000).
[39] See Donald Cardwell, *Wheels, Clocks, and Rockets: A History of Technology* (New York: W. W. Norton, 1995). Cardwell documents the strategic advances in technology that were made during the medieval period.
[40] They built on the medieval heritage mediated through men like Nicholas of Cusa (1401-1464), Leonardo da Vinci (1452-1519), Niccolo Tartaglia (ca. 1499-1557), Geronimo Cardano (1501-1576), Rafael Bombelli (ca. 1526-1573), François Viète (1540-1603), Simon Stevin (1548-1620), Galileo Galilei (1564-1642), Johannes Kelper (1571-1630), and Evangelista Torricelli (1608-1647).

gens (1629-1695), John Wallis (1616-1703), Sir Isaac Newton (1642-1727), and Gottfried Wilhelm Leibniz (1646-1716).[41]

From the late 17th and lasting throughout the 18th century, the calculus was further refined by Jakob Bernoulli (1654-1705), Johann Bernoulli (1667-1748), Michel Rolle (1652-1719), Brook Taylor (1685-1731), Colin Maclaurin (1698-1746), Leonhard Euler (1707-1783), Jean Le Rond d'Alembert (1717-1783), and Joseph-Louis Langrange (1736-1813). In the 19th century, Bernhard Bolzano (1781-1848), Augustin-Louis Cauchy (1789-1857), Karl Weierstrass (1815-1897), Georg Friedrich Riemann (1826-1866), and Julius Wilhelm Richard Dedekind (1831-1916) made final clarifications (in the logical and rigorous sense).

Both the derivative and the integral are founded upon the limit concept, namely the *convergence of an infinite series to a limiting value* (also first studied in the medieval era).[42] It is in the rudimentary principles of the calculus that the *limit* subdues *infinity*. Originally fully articulated in the 17th century (with its philosophical underpinnings secured by Biblical theology), mathematicians since then have been exploring its implications. A few basic abstract principles (the real number continuum, the continuous function concept, coordinate geometry, the limit concept, instantaneous velocity, and area under a curve) have generated a plethora of scientific and technical applications (from airplane construction to rockets shot to the moon and back). The success of the calculus, in the words of David Berlinski, "is among the miracles of mankind."[43]

The "One and the Many"

In this concluding section, I want to analyze how the Biblical worldview specifically speaks to and makes sense of the methods of the calculus.[44] The real number continuum is the fundamental numerical basis for understanding not only the calculus, but also any quantitative analysis of God's created order. It is upon the real number continuum that the function concept rests (a generalized arithmetic in the guise of symbolic algebra). A function is described in mathematical symbols by a formula in terms of dependent (range) and independent (domain) variables. We can get a geometric "feel" for a formula by graphing it on the Cartesian coordinate plane. This grid, consisting of two axes reflecting the real number continuum (the x-axis the real number domain and the y-axis the real number range), unites form (shape) and number. This proximate union of form and number reflects on the way the infinite, triune God (the ultimate One and the Many) has structured both the physical creation and man's intellectual capacities. Man's mind, by creative design, is geared toward unifying diverse aspects of the created reality; i.e., to find the general or unifying principle from the specific or particular. The Cartesian coordinate system, in which form and number are united, is one of many examples of this "unity in diversity" or the "one and the many" principle. It is in the particular world of God's creation (mind and matter, space and time, the invisible and the visible) that mathematics is not so much applied as *revealed.*

It was through the study of motion that the differential calculus found its first illustration. Given a position function (change of distance in terms of time), the first derivative generates a velocity function (change of speed in terms of time). Then it was discovered that velocity *is an aspect of curvature*. The graph of a position

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[41] Both Newton and Leibniz (pronounced "liebnits") are considered to be the immediate co-founders of the calculus. Newton was the first to "put the ideas together" while Leibniz was the first to publish his ideas. The great controversy of their time was over who was to receive the honor of priority in the founding of the calculus. Newton's dynamical world view expressed itself when he spoke of a variable mathematical quantity as a fluent (Latin for "flow") and its rate of change (or derivative) as a fluxion. These two words were first used three hundred years earlier, in the same quantitative context, by the medieval "Calculator" Richard Suiseth (see Boyer, 79). Today, mathematicians use the notation invented by Leibniz, ds/dt for the derivative and ∫ for the integral. This integral sign, an elongated *S*, represents sum (Latin: summa). Leibniz originally named the integral calculus *calculus summatorius*. Joseph-Louis Lagrange (1736-1813) introduced the prime notation for the derivative (e.g., s′). To Lagrange also goes the credit for the adoption of the words "derivative" and "differential."

[42] Although the medieval schoolmen never defined instantaneous velocity as the limit of a ratio (Clagett, 214).

[43] Berlinksi, xiii.

[44] I credit David Berlinski's *A Tour of the Calculus* for many of the ideational connections illustrated in this section.

function makes "visible" instantaneous velocity (the slope of the tangent line at any point on the curve). It is the derivative that unifies two diverse aspects: velocity and curvature.

After investigating this revelation, mathematicians then considered the position function as a *general* concept. Instead of viewing it in particular as a change in distance in terms of time, they viewed it universally as a change in *any* dependent variable in terms of an independent variable (coordinated by the function concept). From this generalization concept came a plenitude of further illustrations. Here again, in the differential calculus, the "unity in diversity" principle is revealed. First, in the derivative that unifies velocity and curvature. Second, one principle, a generalized "position" function, now located in a diversity of characterizations.

Instantaneous velocity is defined as the derivative of a function and area as the integral of a function. The derivative and the integral make use of the limit concept, which was finally given a rigorous definition by the German mathematician Karl Weierstrass (1815-1897) in 1854.[45] In making possible the definition of the derivative, the limit concept unifies two diverse aspects of experience: the *discrete* and the *continuous*. Discrete, Latin for separate, refers to a situation where possibilities are distinct and separate from each other. For example, the number of people in a town is discrete; you cannot have a fraction of a person. Measurements of time and distance, as we have noted, are not discrete because they can very over a continuous range. The position function reflects a *continuous* process (the position of a moving object over time). The limit of a function is a limit of an infinite series taken to *infinity*. The limit is *discrete;* it is a distinct number (whether as the convergence to a specific number in the derivative or the aggregate area in the integral). The limit of a function is the convergence of infinitely many numbers. The limit illustrates the instantaneous (or *continuous*) way in which an object is changing its position. The concept of limit, a concept that defied comprehensive definition until 1854 (*over 2,000 years after* Zeno's brainteasers first cried out for its explication), is another revelation of the "one and the many" principle.

Like addition to subtraction, multiplication to division, and raising a power to extraction of roots, differentiation is the inverse operation of integration. The derivative and integral are two sides of the same coin. Both differentiation and integration characterize diverse realms (instantaneous velocity and area under a

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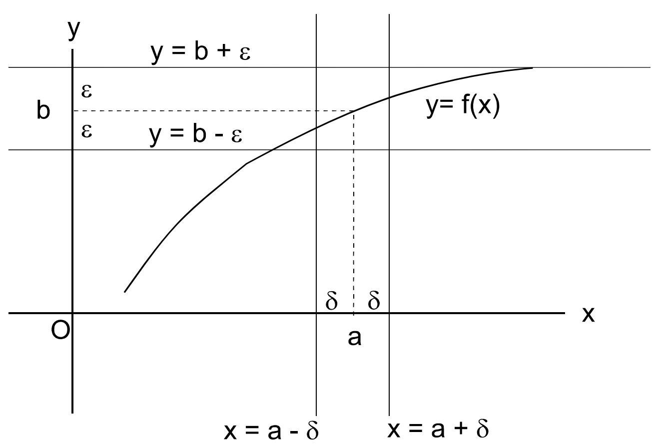
[45] The formal definition of limit developed by Weierstrass and presented to every first-year calculus student is as follows: Give a function $y = f(x)$, then:

$$\lim_{x \to a} f(x) = b$$



means for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if *x* differs from *a* by less than δ, then $f(x)$ differs from *b* by less than ε. Using more esoteric mathematical symbolism: $\forall \varepsilon > 0, \exists \delta > 0 \ni$ if $|x - a| < \delta$, then $|f(x) - b| < \varepsilon$. In other words, for every challenge $\varepsilon > 0$ there is a response $\delta > 0$ that meets a specific condition. In layman's terms, given $f(x) = x^2$, $f(x)$ approaches 9 as a limit (the challenge) as x approaches 3 (the response). It has been said that since 1854 (when the limit was so defined), *only three students have understood it.*

curve). In this context, both are independent of each other. Although independent of each other, there are connected in their dependence upon the limit concept.

The position function gives us a "sense" of place. The velocity function (the derivative of the position function) gives us a "sense" of differences of position. On the Cartesian coordinate grid, this "sense" is pictured as curvature (the slope of the tangent to the point on the curve). Amazingly, the definite integral[46] calculates the area under this curve from position *a* to position *b!* This area is also equal to the distance traveled from position *a* to position *b*. In other words, *distance and area are one number*. This connection should stun and awe the beholder because distance is essentially a one-dimensional concept. Area is a two-dimensional concept. That both should be the same in the calculus is almost too dazzling to digest!

Note the conceptual coherence. The derivative of position is velocity (the measure of how fast an object has gone by). The integral of velocity is distance, the measure of how far an object has been going that fast! It is the Fundamental Theorem of the calculus (afterwards denoted as FTC) that unveils this remarkable and astounding coherence (unity in diversity) that might otherwise go unnoticed.

Note the range of influence revealed by the FTC. It proceeds from the particular to the global (or universal), from diversity to unity, from the many to the one. The derivative of a function converges to a discrete single number, a local particularity. The integral of a function aggregates to a continuous region in space, a global (or unified) vista. According to the FTC, if we are given a continuous global function, then we can apply anti-differentiation and recover the local particular function. Reversing the procedure, given a local continuous function, we can differentiate it and determine the global function. From the derivative to the integral and from the integral to the derivative, from the many (the particular) to the one (global) and from the one (global) to the many (the particular), is the mode revealed by the FTC.

The Fundamental Theorem of the Calculus

This connection was first noted by Isaac Barrow (1630-1677), Newton's mentor. Augustin-Louis Cauchy (1789-1857) presented the first rigorous demonstration of the theorem. Let f(x) be a given function (in our falling stone example, it is the position function) and $f'(x)$ be the derived function or the derivative (in our example, it is the velocity function). The Fundamental theorem of the calculus states that:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

This means that to calculate the area under the curve defined by $f'(x)$ between the bounds *a* and *b*, all we need to do is to find the anti-derivative of $f'(x)$, namely f(x) and subtract f(a) from f(b). Applying this to our falling stone example, given $v = f'(t) = 32t$, then $s = f(t) = 16t^2$. Let a = 1 and b = 5. Then, the area under the curve $f'(t) = 32t$ between *1* and *5* is calculated as follows:

$$\int_1^5 32t dt = f(b) - f(a) = 16(5^2) - 16(1^2) = 400 - 16 = 384$$

This interplay between the one and the many is a fundamental metaphysical conundrum that, in the history of the philosophy of mathematics (in fact, in the history of philosophy), finds no satisfactory resolution. The solution to this interplay finds its true accord *only in the Biblical worldview*.[47] In the triune Godhead, both the one and the many are equally ultimate. In the unity of the Godhead,[48] there are three persons, coeternal

[46] The definite integral calculates the area under a curve from an initial point to a terminal point. Anti-differentiation (the indefinite integral) simply returns the "primitive" function whose derivative is the given function or the function to be integrated.

[47] And this is where the thought of Rousas John Rushdoony, building upon the pioneering work of Cornelius Van Til (1895-1987), finds unexpected, yet remarkable fruition.

[48] Deuteronomy 6:4 states that "God is one." The Hebrew word translated as "one" is transliterated *echad*. This word does not refer to mathematical singularity (as the Hebrew word transliterated *yachid* does); it denotes a compound or collective unity (i.e., a unity of persons).

and coequal, all three fully God, but each distinct in existence. The three distinct persons (or particulars) are the Father, the Son, and the Holy Spirit. All the "particulars" of the Godhead are related to the "universal" and the "universal" is fully expressed in the "particulars." The Father is fully God, the Son is fully God, and the Holy Spirit is fully God. In this sense we can say that the many and the one of the Godhead are both *equally ultimate*. When the triune God creates, He externalizes His ontological nature in what He makes (both the invisible and visible realms). Creation is a temporal or proximate revelation of the eternal "One and the Many." The FTC reveals a proximate and equally ultimate interplay between the "one and the many" of the derivative and the integral.

The deepest purposes of the fundamental connections in mathematics (the proximate one and the many) of which the FTC is one example does not generate the patterned order or laws of the created reality. These connections serve the purpose of allowing these patterns (which are faint echoes of God's creative and sustaining power) *to be revealed*.

I would like to explore one more connection. As we have noted, the derivative historically begin with the analysis of a *concrete* or physical situation; e.g., a falling stone. Beginning with the concrete (the position function), differentiation ends with an *abstract* mathematical object. This object is the derivative of the position function (the velocity function), a real value function from which we can calculate a quantity called instantaneous velocity. Contrariwise, integration begins with an *abstract* mathematical object (the area under a curve) and ends with a *concrete* revelation (the distance traveled by a falling stone).

This balanced "give and take" between the abstract and the concrete reflects upon the nature of the calculus (and the nature of mathematics in general) at the deepest level. From the patterned structures of God's created reality, we can use our minds, created in His image, to formulate abstract ideas. By the methods and tools that mathematics furnishes us, we can discover or unveil the wisdom of God in Christ hidden in the creation. We can then use our conclusions to either increase (1) our appreciation of the beauty of mathematics in how it reveals the proximate "one and the many" of the created realm or (2) the beneficial impact of mathematics upon culture via improved technology (i.e., dominion mandate applications).

Useful Resources for Learning the Calculus

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